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Bistable Systems of Differential Equations with Applications to Tunnel Diode Circuits*

Abstract: A mathematical analysis is developed for nonlinear circuits which have at least two stable steady states, and therefore are of interest as computing or memory elements. Circuits containing one or two tunnel diodes will be analyzed in detail as applications of the theory.

The method is based on the study of a certain "potential function" whose extrema are the steady states of the circuit and whose minima correspond to the stable switching states. This study leads to a qualitative description of all solutions in the large and results in quantitative restrictions on the parameters (*R, L, C* and nonlinear characteristics) which seem of practical importance.

1. Introduction

A basic problem of computing machines is the construction of elements which produce the well-known flip-flop effect. This means essentially that such elements, which are usually electrical circuits, have two stable steady states which represent two binary digits. Such circuits give rise to challenging mathematical problems in the field of nonlinear ordinary differential equations. It seems desirable to develop a mathematical analysis for such nonlinear circuits to get a better understanding of their operation and, possibly, to indicate how to choose parameters and nonlinear characteristics in the optimal way.

The purpose of this paper is to present a method which can be used for such a mathematical analysis. In the following, only the general principle of the method will be given. A further exploitation of this method for estimating response time and quantitative description of the solutions is under way but will not be undertaken here.

I want to express my gratitude to W. Miranker and W. Mayeda for explaining the problem to me and for many stimulating discussions. Miranker also developed a theory of his own which is of asymptotic nature.[1] R. Brayton and R. Willoughby carried out calculations and extensions of these problems.[2]

The circuits to be discussed contain one or two tunnel diodes whose typical feature is described by the characteristic curve shown in Fig. 1. The fact that this curve has a region of negative slope, or "negative resistance," makes the construction of a bistable circuit possible, as will be seen later.

The twin circuit described by Goto[3] contains two tunnel diodes, as shown in Fig. 2. It turns out that for appropriate choice of the parameters E, L, R, C_1, C_2 this circuit is bistable. There are two steady states: one in which most of the energy is in the first diode, and another in which most of the energy is in the second diode. There is a third steady state in which both diodes have the same energy; this state turns out to be unstable.

During the operation of the circuit the voltage E will be increased from 0 to an end-value, and during this process the solution will fall into one of the two possible steady states. In the following discussion we will ignore the time dependence of E and consider E as a parameter in the equation. For small values of E one has only one steady state, which at some critical value of E bifurcates into three stable states.

If one omits in the above circuit one of the diodes, i.e., the capacitance C_2 and f_2, then one has the single diode circuit, which also will be discussed.

Instead of explaining the phenomena in terms of the Goto circuit, we suggest a mechanical analogue which contains in fact all essential features of bistability.

Consider a curve of the shape shown in Fig. 3, e.g., $y=x^4-2x^2-Ex$, where the above graph refers to $E=0$. On this surface we imagine a mass point sliding under the influence of gravity and friction. It is clear that the two minima represent two stable steady states while the maximum is an unstable steady state.

As E increases the above curve becomes asymmetric, and for $E=E_0=8/\sqrt{27}$, the left minimum and the maximum grow together and disappear.

*This paper is a revised and shortened version of IBM Research Report RC-306, August 1, 1960.

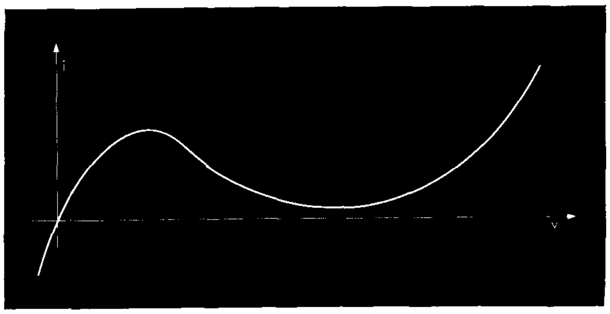


Figure 1

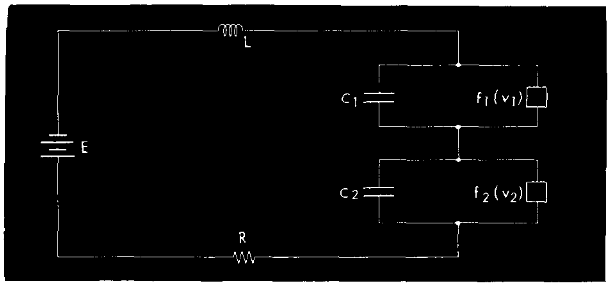


Figure 2

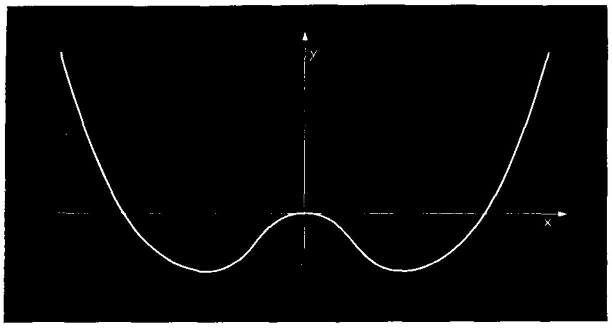


Figure 3

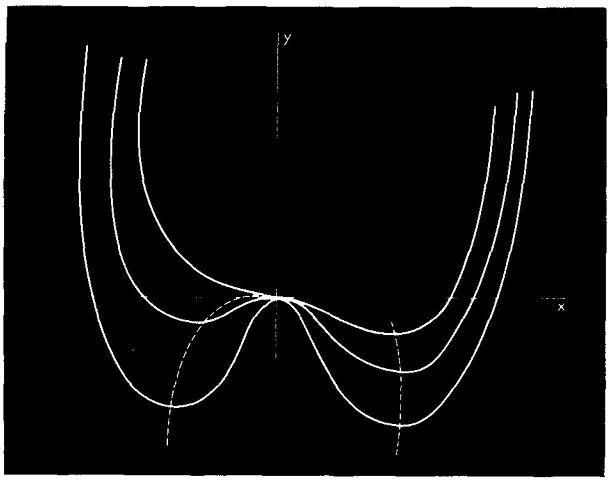


Figure 4

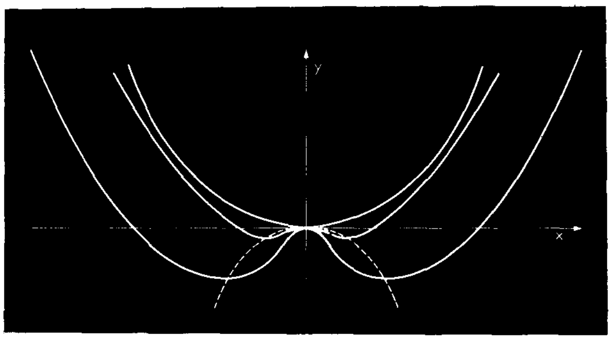


Figure 5

The operation of the single diode circuit can – in this analogy – be described as follows: Choose $E = E_0(1-\delta)$, where δ is a small positive number, so that the left minimum and the maximum are close together (Fig. 4).

If one now supplies a signal by adding to $E_0(1-\delta)$ the value $\pm E_1 > \delta E_0$, then the mass point which initially was assumed to be in the left trough will remain there if the signal is $-E_1$, but will run over to the right if the signal is $+E_1$.

The second analogue which we will need is described by a mass point sliding on the curve $y = x^4 - Ex^2$, which for $E<0$ has only one minimum at $x=0$. As E increases from $E=0$ three extreme points bifurcate from $x=0$, namely, two minima at $x = \pm\sqrt{E/2}$ and a maximum at $x=0$, indicated in Fig. 5. A signal E_1 can be supplied by tilting the whole curves by a small angle E_1. If one starts with $E=0$ so that the mass point will rest at the origin then with increasing values of E the mass point will fall into one of the minima, depending on the sign of E_1.

These two analogues do not only illustrate the bistable character of a circuit but also the amplification which is an essential function of a computer device. The signal may be a very small quantity but the resulting possible end positions of the mass point will be far apart. Mathematically this amplification has its reason in the discontinuous behavior of the number of extreme points in dependence on E.

Intuitively it is clear that the second of the above mechanical analogues will be much more sensitive and will be able to react to smaller signals than the first, since in the first analogue $|E_1|$ had to be bigger than δE_0. In the second analogue such a requirement on E_1 is not needed. In fact, the difference in the two analogues is understood by following the extreme points for different values of E. In the first example two stationary points come together as E approaches the critical value $E = 8/\sqrt{27}$ while the third – a minimum – is at a different position. In the second example all three stationary points come together as $E \to +0$.

This difference gives rise to a classification of bistable systems: Those systems for which at the state of bifurcation only two steady state solutions come together will be called systems of the *first kind;* if three steady state solutions come together we refer to them as systems of the *second kind.*

Obviously bistable systems of the second kind provide a better amplification than systems of the first kind. It will be shown that the parametron and the twin circuit, indeed, belong to the second kind while the single-diode circuit represents a system of the first kind.[4]

While circuits of the second kind are candidates for computer elements, circuits of the first kind probably can only be used for memory devices (which do not require a strong amplification).

For actual computer elements the question of the speed with which the final state is reached is resolved by finding the response time for a circuit. The above analogues serve to illustrate an answer to this question: *To reach the final state in a short time one should have a strong friction and a big curvature at the maximum.* It is hoped that the methods developed here can be used for explicit estimates of response times; however, this will not be attempted in this paper.

The above models can then be considered as representative for the phenomena to be described. In fact, the twin circuits and the parametron can be considered an ingenious realization of the bifurcation of two or three steady states. On the other hand, the theory developed in Section 3 will show that those circuits also are comprised within the frame of the models. This will be done by construction of a "potential function" for the circuits which in the analogues is the total energy.

The analogues illustrate another fact: The bistable character of a circuit, that is, the existence of two stable steady states, implies the existence of an unstable steady state. This corresponds to the geometrical (or topological) assertion: *A positive function in several variables which has two local minima has at least a third extremum which is a saddle point or a maximum.*

Therefore for the construction of bistable circuits it is essential to produce *unstable* steady states. In the twin circuit, for instance, this is achieved by the negative resistance of the characteristic $i=f(v)$, while in the parametron a resonance phenomenon is used.

2. Three circuits

- *The single diode circuit*

This circuit is described in Fig. 6. Here the square is the symbol for the nonlinear characteristic, given by $i=f(v)$. If i is the current and v the voltage in the circuit, then

$$L\frac{di}{dt}=E-Ri-v$$
$$C\frac{dv}{dt}=i-f(v). \tag{2.1}$$

In these equations C is allowed to be a function of v and L can be a function of i. However, in the following we discuss mainly the cases where C, L are constants, except in Section 3 under *Nonlinear capacitance*, page 233.

- *The twin circuit*

This circuit (Fig. 2) contains two tunnel diodes. If v_1, v_2 are the voltage drops across the first and second diodes, respectively, and i the current in the circuit, then one has the equations

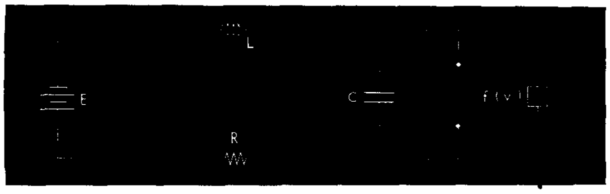


Figure 6

$$L\frac{di}{dt}=E-Ri-v_1-v_2$$
$$C_k\frac{dv_k}{dt}=i-f_k(v_k) \qquad (k=1,2). \tag{2.2}$$

- *The twin circuit with bias*

The above description of the twin circuit is actually a simplification. In Fig. 7 we give a more realistic description which also contains the signal E_B. It will be shown, however, that the corresponding equations can be reduced completely to (2.2).

The equations for this circuit are:

$$L\frac{di}{dt}=\left(E-\frac{R_2E_B}{R_2+R_B}\right)-\left(R_1+\frac{R_2R_B}{R_2+R_B}\right)i-v_1-\frac{R_B}{R_2+R_B}v_2$$
$$C_1\frac{dv_1}{dt}=i-f_1(v_1) \tag{2.3}$$
$$C_2\frac{dv_2}{dt}=\frac{R_B}{R_2+R_B}i+\frac{E_B-v_2}{R_2+R_B}-f_2.$$

These equations can be reduced to the form (2.2) if one sets

$$\hat{v}_1=v_1 \quad , \quad \hat{v}_2=\frac{R_B}{R_2+R_B}v_2$$
$$\hat{f}_1=f_1 \quad , \quad \hat{f}_2=\left(1+\frac{R_2}{R_B}\right)f_2+\frac{v_2-E_B}{R_B}$$
$$\hat{R}=R_1+\frac{R_2R_B}{R_2+R_B},$$
$$\hat{C}_2=C_2\left(1+\frac{R_2}{R_B}\right)^2,$$
$$\hat{E}=E-E_B\frac{R_2}{R_2+R_B}.$$

Then (2.3) takes the form

$$L\frac{di}{dt}=\hat{E}-\hat{R}i-\hat{v}_1-\hat{v}_2$$

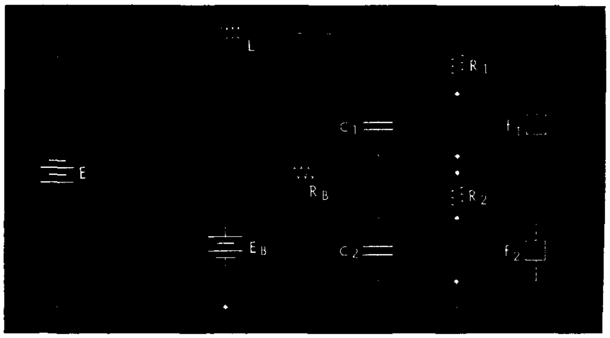


Figure 7

$$\hat{C}_1 \frac{d\hat{v}_1}{dt} = i - \hat{f}_1$$

$$\hat{C}_2 \frac{d\hat{v}_2}{dt} = i - \hat{f}_2 .$$

Therefore, the effect of E_B can be interpreted as a modification of $f_2(v_2)$: The independent variable is stretched according to

$$\hat{v}_2 = \frac{R_B}{R_2+R_B} v_2$$

and a linear function is added to f_2:

$$\hat{f}_2 = \left(1+\frac{R_2}{R_B}\right) f_2 + \frac{v_2-E_B}{R_B} .$$

For $R_B \to \infty$ one has the twin circuit discussed under (2.2).

From now on this circuit will be dropped since it is – at least formally – included in (2.2). The influence of E_B will be interpreted as an asymmetry in the two diodes.

- *The potential function*

The theory developed in the following section is based on the observation that the differential equations (2.1), (2.2), (2.3) can be written in the form

$$L \frac{di}{dt} = \frac{\partial P}{\partial i}$$
$$C_k \frac{dv_k}{dt} = -\frac{\partial P}{\partial v_k} . \qquad (2.4)$$

This holds even for the complicated differential equations (2.3) in which case P has the form

$$P = \hat{E}i - \frac{\hat{R}}{2} i^2 - iv_1 - \frac{R_B}{R_2+R_B} iv_2 + \int f_1 dv_1 + \int f_2 dv_2 + \frac{(v_2-E_B)^2}{2(R_2+R_B)} . \qquad (2.5)$$

In this case v_2 in (2.4) has to be replaced by $\hat{v}_2$. The equations (2.4) are valid even if $L=L(i)$; $C_k=C_k(v_k)$. The function P which has the dimension of a power suffices to describe the differential equations (2.4).

In order to show that the structure (2.4) seems to be typical for electrical circuits we set up the differential equations for the circuit which is obtained from the example of the twin diode with bias by inserting an inductance L_B between E_B and R_B. This circuit gives rise to four first-order equations.

Denoting the current through L and L_B by i and j and the voltages across the diodes by v_1, v_2 we find

$$L \frac{di}{dt} = \frac{\partial P}{\partial i}$$

$$L_B \frac{dj}{dt} = \frac{\partial P}{\partial j}$$

$$C_1 \frac{dv_1}{dt} = -\frac{\partial P}{\partial v_1}$$

$$C_2 \frac{dv_2}{dt} = -\frac{\partial P}{\partial v_2} ,$$

where

$$P = Ei + E_B j - \left\{(R_1+R_2)\frac{i^2}{2} + R_2 ij + (R_2+R_B)\frac{j^2}{2}\right\} - i(v_1+v_2) - jv_2 + \int_0^{v_1} f_1 dv_1 + \int_0^{v_2} f_2(v_2) dv_2 .$$

The equations (2.4) are obtained from these by setting $L_B=0$. In the following[5] we will work only with the equations (2.4).

3. A general method

- *The potential function*

In this Section we study systems of the form

$$L \frac{di}{dt} = E - Ri - \sum_{k=1}^{n} v_k$$
$$C_k \frac{dv_k}{dt} = i - f_k(v_k) \qquad (k=1, 2, \ldots, n), \qquad (3.1)$$

where we consider E as a fixed parameter, i.e., independent of t. As we saw previously this covers the circuits under discussion.

It is our main purpose here to find conditions which ensure a bistable situation of the system, for which, moreover, all solutions tend to steady state solutions. It is well known that nonlinear systems need not have this property, and that there might be bounded solutions which never approach a steady state. In fact, van der Pol's equation which admits a limit cycle, i.e., self-sustained oscillations, describes a phenomenon of this kind.

It is fortunate that the equations (3.1) allow for very simple conditions which guarantee that all solutions tend to steady states, as will be shown in this section. The method employed is based on an idea of Liapounov to construct a function which decreases along solutions with increasing time. While usually the energy can be used as such a function this is not possible for circuits with negative resistance. For the construction of such a

Liapounov function the potential function P will be of basic importance. Moreover, the investigation of the stability of the solution is particularly simple.

Setting

$$I(i, v)=\frac{1}{L}\left(E-Ri-\sum_{k=1}^{n} v_k\right)$$

$$V_k(i, v)=\frac{1}{C_k}(i-f_k), \tag{3.2}$$

the above system takes the form

$$\frac{di}{dt}=I(i, v) \quad ; \quad \frac{dv_k}{dt}=V_k(i, v) \qquad (k=1, 2, \ldots, n). \tag{3.1*}$$

Observing that the coefficients of v_k in LI are -1 and the coefficient of i in C_kV_k is $+1$ we see that

$$\frac{\partial}{\partial v_k}(LI)=-\frac{\partial}{\partial i}(C_kV_k).$$

This, together with the fact that V_k is independent of $v_l(l\neq k)$ shows that the integral

$$P(i, v)=\int_{(0,0)}^{(i,v)} LIdi-\sum_{k=1}^{n} C_kV_kdv_k \tag{3.3}$$

taken as a line integral in the $n+1$ dimensional $i, v_1, \ldots, v_n$ space is independent of the path. Hence integrating from $i=v_1=\ldots=v_n=0$ to $(i, v_1, \ldots, v_n)$ along any path P is a function of the end point only.

This quantity P has the dimension of a power (i.e., voltage times current) and is attached to any particular state of the circuit, no matter how it was brought into this state. This concept is reminiscent of the concept of potential energy and ought to be of physical significance. From the mathematical point of view the importance of the function $P(i, v)$ is put into evidence by the remark that *the differential equations (3.1) are completely determined by the function P*. In fact, from (3.3) it follows that

$$\frac{\partial P}{\partial i}=LI \quad ; \quad -\frac{\partial P}{\partial v_k}=C_kV_k, \tag{3.4}$$

and (3.1) takes the form

$$L\frac{di}{dt}=\frac{\partial P}{\partial i} \quad ; \quad C_k\frac{dv_k}{dt}=-\frac{\partial P}{\partial v_k}. \tag{3.1**}$$

Moreover, the extrema (critical points) of P are given by

$$\frac{\partial P}{\partial i}=\frac{\partial P}{\partial v_k}=0 \qquad (k=1, 2, \ldots, n)$$

and coincide, by (3.4), with the steady state solutions of the system of differential equations (3.1).

For the following it will be useful to have explicit expressions for P. Two forms prove especially appropriate: Direct integration of (3.3) using (3.2) gives

$$P=Ei-\frac{Ri^2}{2}-i\sum_{k=1}^{n} v_k+\sum_{k=1}^{n} F_k(v_k), \tag{3.5}$$

where $F_k(v_k)$ are the integrals of the characteristic nonlinearities $f_k(v_k)$, that is,

$$F_k(v_k)=\int_0^{v_k} f_k(\lambda)d\lambda.$$

Secondly, if one uses $I, v_1, \ldots, v_n$ instead of $i, v_1, \ldots, v_n$ as independent variables one finds

$$P=\frac{-L^2I^2}{2R}+U(v), \tag{3.6}$$

where the function

$$U(v)=\frac{\left(E-\sum_{k=1}^{n} v_k\right)^2}{2R}+\sum_{k=1}^{n} F_k(v_k) \tag{3.7}$$

is independent of I. The functions P and $U(v)=P/_{I=0}$ will be referred to as potential functions.

The formula (3.6) can be derived from (3.5) but one has an easier approach if one integrates (3.3) by parts:

$$P=LIi-\int(LidI+\Sigma C_kV_kdv_k). \tag{3.3*}$$

Using $I, v_1, \ldots, v_n$ as independent variables one has

$$\frac{\partial i}{\partial I}=-\frac{L}{R} \quad , \quad \frac{\partial i}{\partial v_k}=-\frac{1}{R}$$

and, therefore, from (3.3*)

$$\frac{\partial P}{\partial I}=LI\frac{\partial i}{\partial I}=-\frac{L^2}{R}I$$

$$\frac{\partial P}{\partial v_k}=LI\frac{\partial i}{\partial v_k}-C_kV_k=-\frac{E-\Sigma v_k}{R}+f_k(v_k).$$

This shows that $\partial P/\partial v_k$ is independent of I, and integration yields (3.6). The integration constant corresponds to the normalization $P=0$ for $i=v=0$, or $LI=E$, $v_k=0$, but this is irrelevant.

- *The S-function*

For the following we use one more function:

$$Q=\frac{1}{2}LI^2+\frac{1}{2}\sum_{k=1}^{n} C_kV_k^2, \tag{3.8}$$

which vanishes only on the steady solutions and is positive otherwise. If one evaluates Q on a solution one finds for the t derivative

$$-\frac{d}{dt}Q=-\frac{\partial Q}{\partial i}I-\sum_{k=1}^{n}\frac{\partial Q}{\partial v_k}V_k$$

$$-\frac{d}{dt}Q=RI^2+\sum_{k=1}^{n} f_k'V_k^2. \tag{3.9}$$

If one would assume that $f_k(v_k)$ are increasing functions, as is the case for ordinary diodes, then the right-hand side would be positive if one excludes the steady states and Q would be a decreasing function and tend to zero as $t\to\infty$. This proves that all solutions tend to steady states if $f_k'(v_k)>0$.

The novel feature of the tunnel diodes, however, is the fact that f'_k is negative in some interval (negative resistance). It is this fact which permits bistable behavior.

To take into account characteristics with negative slope we form

$$S=Q+\lambda P, \tag{3.10}$$

where P, Q are the functions defined in (3.3) and (3.8) and λ is a positive constant to be chosen appropriately. An easy calculation [using (3.3) and (3.9)] gives

$$-\frac{d}{dt}S=(R-\lambda L)I^2+\sum_{k=1}^{n}(f'_k+\lambda C_k)V_k^2. \tag{3.11}$$

To ensure that the right-hand side is positive we choose λ in the interval

$$\frac{-f'_k}{C_k}<\lambda<\frac{R}{L} \tag{3.11*}$$

assuming here that

$$\frac{f'_k}{C_k}+\frac{R}{L}>0. \tag{3.12}$$

The last inequality is a restriction on f'_k. The slope f'_k is allowed to be negative but should remain $>-C_kR/L$. In the existing circuit this condition turns out to be satisfied. At a later place we will give further motivation for the need of such a condition to avoid self-sustained oscillations.

We show now

• *Theorem 3.1*

The extreme points (or critical points) of S coincide with the steady states of (3.1), provided (3.11) holds.*

Proof: We calculate the partial derivatives of S

$$\begin{aligned}\frac{\partial S}{\partial i}&=-(R-\lambda L)I+\sum_{k=1}^{n}V_k\\ \frac{\partial S}{\partial v_k}&=-I-(f'_k+\lambda C_k)V_k,\end{aligned} \tag{3.13}$$

and see that the gradient of S vanishes at steady state solutions. By (3.11) the directional deviation dS/dt does not vanish at any other point, which proves the statement.

• *Behavior of the solutions for $t\rightarrow+\infty$*

In order to establish the bistable character of a circuit in the sense that the solutions approach two or more steady states, it is important to exclude the possibility of self-sustained oscillations, which indeed can occur for systems of the form (3.1). From the mathematical point of view the boundedness of the solution also ought to be guaranteed in order to be sure of the existence of the solution for all $t>0$. This last fact seems rather obvious from the practical point of view and we state conditions for boundedness of the solutions without proof.

• *Theorem 3.2*

If

$$\left.\begin{aligned}&xf_k(x)\geq 0 \text{ for all } x \text{ and}\\ &xf_k(x)>\frac{E^2}{R} \text{ for } x>A_k\end{aligned}\right\} \tag{3.14}$$

then all solutions of (3.1) remain bounded.

For a proof we refer to the appendix of Section 3. The above general conditions of f_k will be adopted throughout the following. They are certainly valid for the circuits considered.

The main result, however, is contained in

• *Theorem 3.3*

Under the condition

$$\frac{f'_k}{C_k}+\frac{R}{L}>0$$

all solutions of (3.1) tend to the steady state solutions which are assumed to be finite in number.

Proof: From (3.11) it is seen that S is a decreasing function along the solutions, if λ is chosen according to (3.11*). More precisely: For $\varepsilon>0$ one has outside the neighborhoods

$$I^2+\Sigma V_k^2<\varepsilon \tag{3.15}$$

about the steady states

$$-\frac{dS}{dt}\geq c(I^2+\Sigma V_k^2)\geq c\varepsilon,$$

where

$$c=\mathrm{Min}(R-\lambda L, f'_k+\lambda C_k)>0.$$

Hence

$S\leq S/_{t=0}-c\varepsilon t$. Since, however, S is bounded from below:

$$\begin{aligned}S=Q+\lambda P&\geq(R-\lambda L)\frac{I^2L}{2R}+\lambda U(v)\\ &\geq\sum_{k=1}^{n}F_k(v_k)\geq 0,\end{aligned} \tag{3.16}$$

S cannot decrease indefinitely and for $t\geq S_0/c\varepsilon$ the solution must have penetrated into one of the neighborhoods described by (3.15). In fact, the total time a solution spends outside this neighborhood is at most $c^{-1}\varepsilon^{-1}S_0$. This proves the statement.

• *Stable steady states*

Since all solutions tend to steady state solutions it seems most important to investigate their stability behavior which can be done in two ways:

α) The first method consists in investigating the linearized equations near a steady state and their characteristic exponents which are given as eigenvalues $\alpha_0, \alpha_1, \ldots, \alpha_n$ of the matrix

$$M=\begin{bmatrix} -\frac{R}{L} & -\frac{1}{\sqrt{C_1L}} & \cdots & & -\frac{1}{\sqrt{C_nL}} \\ \frac{1}{\sqrt{C_1L}} & -\frac{f_1'}{C_1} & 0 & \cdot\ \cdot & 0 \\ \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & 0 \\ \frac{1}{\sqrt{C_nL}} & 0 & \cdots & 0 & -\frac{f_n'}{C_n} \end{bmatrix} \tag{3.17}$$

evaluated at the steady state to be considered.

β) The second method uses the fact that the steady states are the extreme values of the function S which decreases along the solutions. From this fact it follows that a local *minimum* of S corresponds to *stable* steady states. This argument goes back to Lagrange and Dirichlet[6] and is well known. Therefore we will try to determine the minima of S.

We remark that S has at least one minimum since S is nonnegative and tends to ∞ as i, v tend to infinity [here we use (3.14)]. This proves *the existence of at least one stable steady state* under the conditions (3.12), (3.14).

To decide whether an extreme point of S is a minimum or not we compute the symmetric matrix of second derivatives of S. For this purpose we introduce

$$x_0=\sqrt{L}i \qquad x_k=\sqrt{C_k}v_k$$

$$X_0=\sqrt{L}I \qquad X_k=\sqrt{C_k}V_k$$

and determine

$$\frac{\partial^2 S}{\partial x_k \partial x_l} = \frac{\partial^2 Q}{\partial x_k \partial x_l} + \lambda \frac{\partial^2 P}{\partial x_k \partial x_l}.$$

Since

$$Q=\frac{1}{2}\sum_{k=0}^{n} X_k^2$$

and

$$\left(\frac{\partial X_k}{\partial x_l}\right)=M$$

one finds

$$\left(\frac{\partial^2 Q}{\partial x_k \partial x_l}\right)=M^TM.$$

The equation (3.3) takes the form

$$P=\int LIdi-\sum_{k=1}^{n} C_kV_kdv_k$$

$$=\int X_0dx_0-\sum_{k=1}^{n} X_kdx_k,$$

from which one finds

$$\frac{\partial P}{\partial x_0}=X_0,\ \frac{\partial P}{\partial x_k}=-X_k$$

which we combine in vector notation to

$$\frac{\partial P}{\partial x}=JX$$

with

$$J=\text{diag}(1,-1,-1,\ldots,-1).$$

Hence

$$\left(\frac{\partial^2 P}{\partial x_k \partial x_l}\right)=JM=N$$

which defines the symmetric matrix

$$N=\begin{bmatrix} -\frac{R}{L} & -\frac{1}{\sqrt{C_1L}} & \cdots & & -\frac{1}{\sqrt{C_nL}} \\ -\frac{1}{\sqrt{C_1L}} & \frac{f_1'}{C_1} & 0 & \cdot\ \cdot & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & 0 & & \cdot & \cdot \\ & \cdot & & \cdot & 0 \\ -\frac{1}{\sqrt{C_nL}} & 0 & \cdots & 0 & \frac{f_n'}{C_n} \end{bmatrix}$$

combining these results we have from $J^TJ=I$

$$\left(\frac{\partial^2 S}{\partial x_k \partial x_l}\right)=M^TM+\lambda N=N^2+\lambda N.$$

Denoting the real eigenvalues of N by $\nu_0,\ldots,\nu_n$ *we have a minimum of S if*

$$\nu_k(\nu_k+\lambda)>0,$$

i.e., if all eigenvalues of N lie outside the interval $(-\lambda, 0)$.

Another sufficient condition for a minimum of S seems more useful and will be exploited in the following Section: With (3.6) and (3.8) S takes the form

$$S=\frac{1}{2}(R-\lambda L)\frac{L}{R}I^2+\frac{1}{2}\Sigma C_kV_k^2+\lambda U(v). \tag{3.18}$$

This formula shows immediately:

● *Theorem 3.4*

The extreme points of S correspond to the extreme points of U. Every local minimum of U represents a local minimum of S and hence a stable steady state of (3.1).

This theorem allows us to reduce the problem by one dimension and therefore simplifies many calculations.

Proof: From (3.18) one computes the identity

$$S_{v_k}+f_k'S_i=-(R-\lambda L)\left(\frac{1}{R}+f_k'\right)I+\lambda U_{v_k}$$

from which it is evident that the points

$$I=0; \qquad U_{v_k}=0$$

yield extreme points of S. Conversely, at extreme points of S one has $I=0$, hence from the above equation $U_{v_k}=0$. This proves the first half of the theorem.

The second part is obvious from (3.14), since at a minimum point $v=v'$ of U and $I=0$,

$$S=\lambda U(v'),$$

whereas for neighboring values of v, I,

$$S\geqslant\lambda U(v)\geqslant\lambda U(v').$$

- *A Legendre transformation*

In the following Section the function $U(v)$ will be studied for various circuits. Here we remark that if in the equations for steady states of (3.1)

$$LI=E-Ri-\Sigma v_k=0$$
$$i-f_k(v_k)=0,$$

the variable i is eliminated, one obtains $I=0$ and

$$+\frac{E-\Sigma v_k}{R}-f_k(v_k)=0,$$

which are precisely the conditions for extreme points of U, i.e.,

$$\frac{\partial U}{\partial v_k}=0.$$

This fact can be understood more clearly if one uses instead of i, v_k the variables I, v_k. These variables, i, v_k and I, v_k are related by a Legendre transformation.[7] Namely, if one uses (3.3*) with

$$K(I,v)=\int LidI+\sum_k C_kV_kdv_k=LIi-P,$$

one has

$$Li=\frac{\partial K}{\partial I}\quad;\quad C_kV_k=\frac{\partial K}{\partial v_k}.$$

This transformation can easily be carried out explicitly since i and I are related by linear equations.

For some purposes it can be useful to go from the variables i, v_k to I, V_k. The steady states correspond exactly to the points $I=V_k=0$. It is clear, however, that in the presence of more than one steady state solution the transformation from the i, v_k to the I, V_k is not one to one since all steady states correspond to one point $I=V_k=0$. Indeed, this transformation is nonlinear and rather complicated. However, if one restricts attention to a domain (in the v_k variables) where U is convex *then the correspondence between (i, v_k) and (I, V_k) is one-to-one* and this transformation can be used with success.

To prove this statement we use (3.6). With I, v_k as independent variables one finds

$$\frac{\partial U}{\partial v_k}=\left.\frac{\partial P}{\partial v_k}\right|_{I=\text{const}}=-C_kV_k-\frac{LI}{R}.$$

Hence, for every given I, V_k the value of the v_k are uniquely determined. This follows from the convexity of U and the fact that for a family of parallel planes there is at most one which touches a convex surface. Indeed $z=U(v_1,\ldots,v_n)$ can be considered a convex surface and the vector

$$\left(\frac{\partial U}{\partial v_1},\ldots,\frac{\partial U}{\partial v_n},-1\right)$$

defines a normal to the surface. Hence for a given I, V_k there is at most one vector v_k. The current i can then be found from the linear equation

$$LI=E-\Sigma v_k-Ri.$$

- *Nonlinear capacitance*

The above results were derived under the tacit assumption that L, C, R were positive constants. This is unrealistic but frequently sufficient for the description of a circuit. For a tunnel diode, however, the capacitance C is a function of a voltage usually approximated by

$$C(v)=C_0\left(1-\frac{v}{G}\right)^{-\frac{1}{2}}.$$

It is interesting that the above method can be modified so as to extend to the equations (3.1) with variable $C_k=C_k(v)>0$.

The formulae derived for the potential function P will generalize without any change. However, the function Q defined by (3.8) when differentiated will contribute new terms arising from differentiation of C_k. To avoid these terms we add to P not Q but a multiple of I^2. It turns out that

$$\tilde{S}=\frac{L^2}{R}I^2+P(i,v)=\frac{L^2}{2R}I^2+U(v)\qquad(3.19)$$

is the appropriate function to work with.[8]

If we carry out the Legendre transformation of i, v into I, v discussed under stable steady states, the differential equations (3.1) go over into

$$\frac{dI}{dt}=-\left(\frac{R}{L}-\frac{1}{cR}\right)I+\frac{1}{L}\sum_{k=1}^{n}C_k^{-1}U_{v_k}$$

$$C_k\frac{dv_k}{dt}=-\frac{L}{R}I-U_{v_k}.\qquad(3.20)$$

With these equations one finds readily

$$-\frac{d\tilde{S}}{dt}=-\frac{L^2I}{R}\frac{dI}{dt}-\sum_k U_{v_k}\frac{dv_k}{dt}$$

$$=\frac{d\tilde{S}}{dt}=\frac{L^2}{R}\left(\frac{R}{L}-\frac{1}{cR}\right)I^2+\Sigma C_k^{-1}U_{v_k}^2,\qquad(3.21)$$

where

$$\frac{1}{C}=\sum_{k=1}^{n}\frac{1}{C_k}.$$

We now make the assumption

$$\frac{1}{R}\Sigma C_k^{-1}=\frac{1}{Rc}<\frac{R}{L}.\qquad(3.22)$$

Then $\tilde{S}$ is a decreasing function of t by (3.21) if one substitutes a solution into $\tilde{S}$.

Furthermore, the stationary points of $\tilde{S}$ are given by

$$\frac{\partial \tilde{S}}{\partial i} = -LI = 0$$

$$\frac{\partial \tilde{S}}{\partial v_k} = -\frac{L}{R} I + U_{v_k} = 0 ,$$

which gives

$$I = U_{v_k} = 0 .$$

Hence the extreme points of $\tilde{S}$ coincide with the steady states of the equation.

Following the same ideas as in the Section on solutions for $t \to +\infty$, we have

• *Theorem 3.5*

Under the condition (3.22):

$$\frac{1}{R} \Sigma \frac{1}{C_k} < \frac{R}{L}$$

all solutions of (3.1) (with nonlinear positive C_k) tend to steady state solutions which are assumed to be finite in number.

• *Theorem 3.6*

The extreme points of $\tilde{S}$ coincide with the extreme points of $U(v)$, $I=0$. Every local minimum of U corresponds to a local minimum of $\tilde{S}$ and vice versa.

The proof of the last theorem follows immediately from (3.19).

The last theorems show that the condition

$$\frac{f_k'}{C_k} + \frac{R}{L} > 0$$

can be replaced by the new condition

$$\frac{1}{CR} < \frac{R}{L}$$

without violating the conclusion. The last condition is satisfied for a simple circuit studied by Esaki where $c(v)$ indeed is assumed to be not a constant. However, in other cases (3.22) is definitely violated and one has to check the condition (3.12) which is a restriction on the characteristic function $f_k(v_k)$. Notice that all above conditions are comparisons of two frequencies which have simple physical interpretations. On the other hand the potential functions $U(v)$ entering Theorems 3.4 and 3.6 are the same and are basic for the following discussion.

Appendix: Proof of Theorem 3.2

We use the energy expression

$$W = \frac{1}{2} Li^2 + \frac{1}{2} \sum_{k=1}^{n} C_k v_k^2$$

of the circuit and restrict attention to large values of W:

$$W > W_0 = \frac{1}{2} L \left(\frac{E}{R}\right) + \frac{1}{2} \Sigma C_k A_k^2 ,$$

where A_k was defined by (3.14). It is clear that this inequality implies that at least one coordinate is large. To make this more explicit we show that one of the following $(n+1)$ relations holds

$$|i| > \frac{E}{R}; \qquad |v_k| > A_k .$$

For, if the first is violated, then it follows from the above

$$\Sigma C_k v_k^2 \quad > \quad \Sigma C_k A_k^2 ,$$

which implies that $|v_k| > A_k$ for at least one k.

If one differentiates W along a solution one finds

$$-\frac{dW}{dt} = Ri^2 - Ei + \sum_{k=1}^{n} f_k v_k$$

$$= Ri\left(i - \frac{E}{R}\right) + \sum_{k=1}^{n} f_k v_k .$$

If $|i| > \frac{E}{R}$ this is obviously nonnegative. But if $|i| \leq \frac{E}{R}$ then $|v_k| > A_k$ for at least one k and by assumption

$\Sigma v_k f_k \geq \frac{E^2}{R}$ so that

$$-\frac{dW}{dt} \geq Ri^2 - Ei + \frac{E^2}{R} \geq Ri^2 \geq 0$$

which proves that

$$-\frac{dW}{dt} \geq 0 \text{ for } W > W_0 .$$

This shows that solutions starting in the bounded domain $W=$const $(>W_0)$ remain inside this domain for all positive t. This proves the theorem and, moreover, gives explicit bounds for the solution.

4. Applications to circuits containing tunnel diodes

• *The single diode circuit*

The equations for the circuit to be discussed here correspond to $n=1$ in (3.1). We may assume immediately that $C=C(v)$ is a function of v.

For the characteristic function $f(v)$ we will assume that f is positive for $v>0$ and $f<0$ for $v<0$, $f(0)=0$. Moreover, the positive v-axis (Fig. 8) decomposes into three intervals where f is alternately increasing, decreasing and increasing, i.e.,

$$f'>0 \text{ in } -\infty<v<a, \quad v>b>a>0$$

$$f'<0 \text{ in } a<v<b.$$

The potential function $U(v)$ has the form

$$U(v) = \frac{(E-v)^2}{2R} + \int f(v)\,dv$$

and its derivative

$$\frac{dU}{dv} = -\frac{E-v}{R} + f(v).$$

Therefore the steady state solutions are obtained as zeros of dU/dv, which geometrically can be constructed (Fig. 9) by intersecting the graph of the curve $i=f(v)$ with the straight line

$$i = \frac{E-v}{R}, \tag{4.0}$$

as is well known. Depending on the values of E, R one will obtain 1, 2 or 3 solutions.

The function $U(v)$ can then be interpreted – up to a constant – as the area between the curve $i=f(v)$ and the straight line. The graph of U for the above situation is of the shape depicted in Fig. 10, which shows that U has two minima and one maximum. Therefore, by the discussion Section 3: If one of the condition (3.22) or (3.12) is satisfied then one has two stable steady states. The third steady state is actually unstable.

To discuss the restrictions on the parameters we make the requirement that three points of intersection will occur for appropriate choice of E, so that switching is possible. This leads to

$$\text{Max}\,(-f') > \frac{1}{R}. \tag{4.1}$$

This inequality shows that the condition (3.12)

$$-\frac{f'}{C} < \frac{R}{L} \text{ for all } v$$

implies

$$\frac{1}{CR} < \frac{1}{C}\,(\text{Max}-f') < \frac{R}{L}$$

and hence implies (3.22). Therefore it is less restrictive to assume only

$$\frac{1}{CR} < \frac{R}{L}. \tag{3.22}$$

Moreover the statements concerning the nonlinear capacitance were derived under this assumption only. Under this condition we have shown that each solution tends to one of these three steady state solutions.

A more precise picture can be obtained by studying the level lines of the function

$$\tilde{S} = \frac{L^2}{2R} I^2 + U(v)$$

which we represent in an I, v plane, Fig. 11. The extreme points occur all on the v axis and we discuss a case where all three points occur.

Since the value of the function $\tilde{S}$ along a solution decreases, every solution cuts these level lines – always in the same sense. Almost all solutions tend to the two stable steady states. The solutions tending to the saddle point of $\tilde{S}$ form two curves which spiral to infinity. The existence of these two solutions is an easy consequence of a theorem of J. Hadamard.[9]

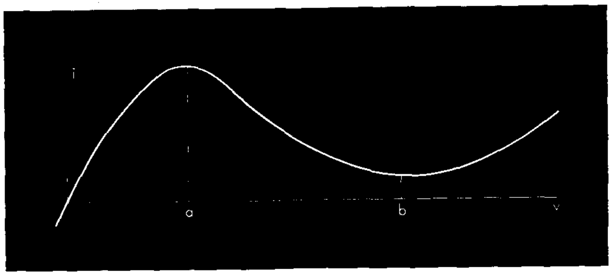


Figure 8

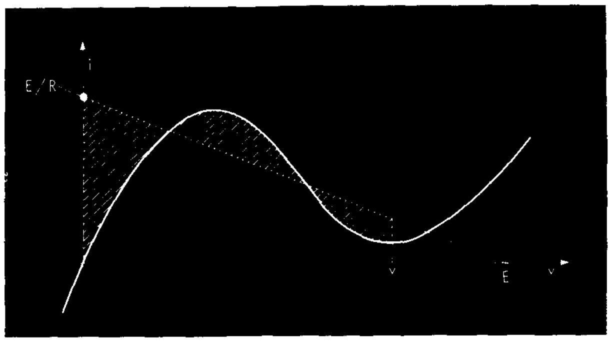


Figure 9

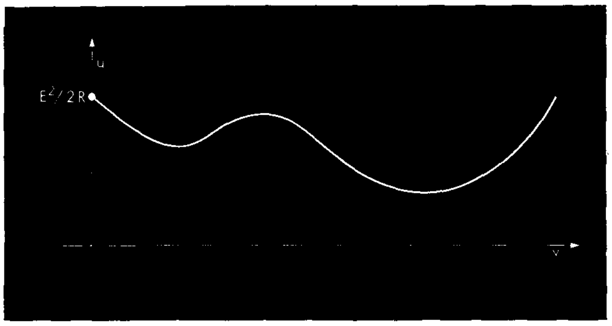


Figure 10

In the case where E is zero or so large that the straight line (4.0) hits the curve only in one point, one will have only one stable steady state, since in this case U and also $\tilde{S}$ have only one extreme point which is a minimum. The question how fast a solution approaches this steady state solution can easily be answered by appropriate estimates of the right-hand side in (3.21).

In this case one can, for instance, estimate

$$\frac{U'^2}{U-U_0} \geq \alpha,$$

where U_0 is the minimum value of U and a positive constant. Then one finds from (3.21) an estimate of the form

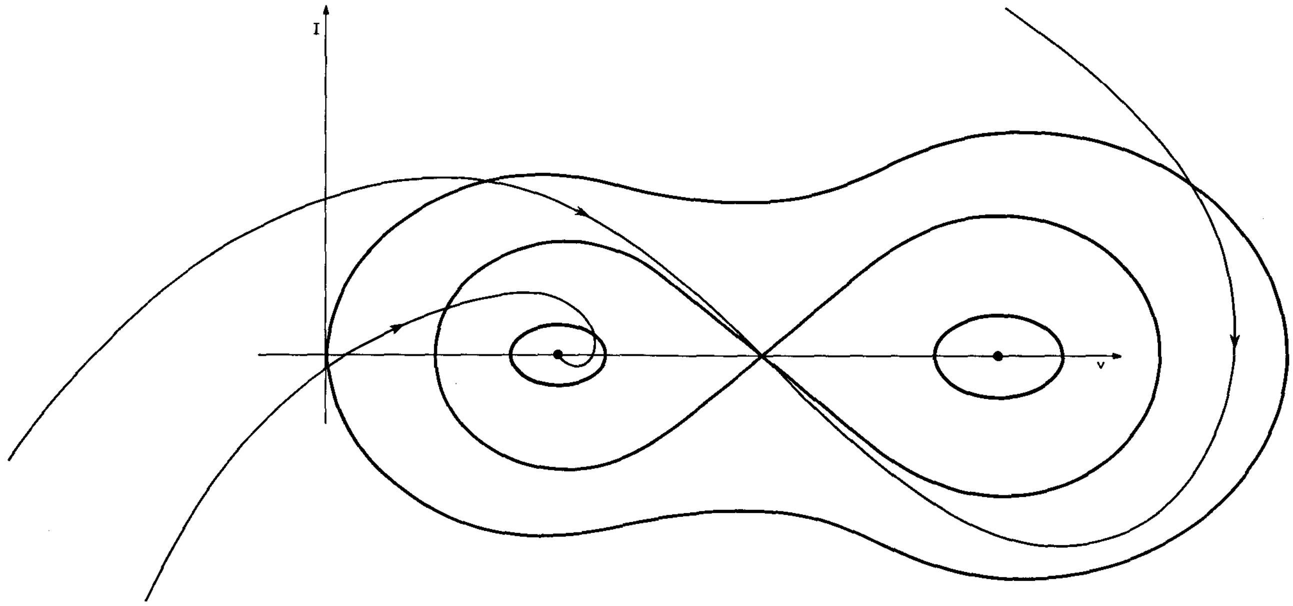


Figure 11

$$-\frac{d\bar{S}}{dt}\geq\alpha(\bar{S}-\bar{S}_0)$$

and α^{-1} represents a certain rseponse time. Refinements of this method are being worked out numerically by R. Brayton.[2]

Finally we discuss the generation and disappearances of steady state solutions as the parameter E varies. For $E=0$ there is one stable steady state only at $v=0$ which increases as E increases. The moment when the line $i=(E-v)/R$ touches $i=f$ *another* steady state occurs which bifurcates into a stable and an unstable one. Finally when E is increased further the first minimum and the saddle points of $\bar{S}$ concur and disappear. The situation can be visualized easily with the graph of U.

The study of the equation with t-dependent $E=E(t)$ will not be undertaken here but requires special attention.

• *Occurrence of limit cycles*

In the previous investigation two frequency relations were required, namely (3.12) and (3.22). We want to show that some frequency restrictions of this type are necessary indeed for the bistable character of the system. We discuss the equation (3.1) for $n=1$ and a constant capacitance C. For the convenience of the argument we assume

$$-f'<\frac{1}{R}, \qquad (4.2)$$

which guarantees the existence of only *one* steady state [reversed inequality of (4.1)]. If one now violates condition (3.12) and assumes

$$\text{Max}\left(-\frac{f'}{c}\right)>\frac{R}{L}, \qquad (4.3)$$

then the argument of Section 3 breaks down since in (3.11) the right-hand side cannot be made positive. In this case, however, we can prove that all solutions tend to one or possibly several limit cycles for $t\to+\infty$, i.e., one has a self-sustained oscillation, if E is chosen so that (4.3) holds at the steady state solution.

For the proof of this statement we remember that all solutions are bounded as was shown in the appendix of Section 3. There exists exactly one steady state solution at which the matrix of the linearized equation is

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ +\frac{1}{C} & -\frac{f'}{C} \end{bmatrix}.$$

The eigenvalues of this matrix, say α_1, α_2, satisfy

$$\alpha_1+\alpha_2=-\left(\frac{f'}{C}+\frac{R}{L}\right)>0$$

and

$$\alpha_1\alpha_2=\frac{1+Rf'}{LC}>0\,.$$

This implies that $Re\alpha_1>0$, $Re\alpha_2>0$ so that all solutions escape from the steady state as $t\to+\infty$. In other words one can construct an ellipse about the steady state solutions in such a manner that the vector field points outwards on this ellipse. The theory of Poincare-Bendixson applies exactly to this situation and guarantees the existence of at least one limit cycle. (We refer to E. A. Coddington and N. Levinson.[10])

The conditions (4.2), (4.3) are of course not exactly the negation of (3.12) or (3.22) and our case distinction is not exhaustive. However, this result shows that one needs some parameter restriction to exclude the occurrence of limit cycles.

The conditions (4.2), (4.3) can be relaxed. It suffices that (4.3) holds for every steady state solution and (4.2) can be omitted completely. Then one can still exhibit the existence of limit cycles.

- *The twin diode*

For discussion of the twin diode circuit (as devised by E. Goto), we start with the equations

$$L\frac{di}{dt}=E-Ri-\sum_{k=1}^{2}v_k$$

$$C_k\frac{dv_k}{dt}=i-f_k(v)\,, \qquad (4.4)$$

i.e., with the equations (3.1) for $n=2$. We omitted the bias (as discussed in Section 2) since those equations can be reduced to (4.4) (see Section 2).

We will assume that the diodes have practically the same characteristic functions

$$f_1(x)=f_2(x) \quad ; \quad C_1=C_2 \qquad (4.5)$$

and later discuss the modifications to be derived from an asymmetry. Such an asymmetry will in fact be produced by a bias (see Section 2, *Twin circuit with bias,* page 228).

Basic for the theory outlined in Section 3 is the function

$$U(v_1, v_2)=\frac{(E-v_1-v_2)^2}{2R}+\int_0^{v_1} f_1(v_1)\,dv_1 + \int_0^{v_2} f_2(v_2)\,dv_2 \qquad (4.6)$$

whose extreme points coincide with the steady state solutions of (4.1). Even though U is only a function of two variables (and not all 3 variables i, v_1, v_2) it is hard to visualize this surface. Indeed the number of critical points will depend very much on the shape of f_1, f_2 and the size of E, R. There might be as many as nine steady solutions under suitable conditions.

Since it is difficult to picture this surface we suggest finding the extreme points only from

$$\frac{\partial U}{\partial v_1}=-\frac{(E-v_1-v_2)}{R}+f_1(v_1)=0$$

$$\frac{\partial U}{\partial v_2}=-\frac{(E-v_1-v_2)}{R}+f_2(v_2)=0\,. \qquad (4.6a)$$

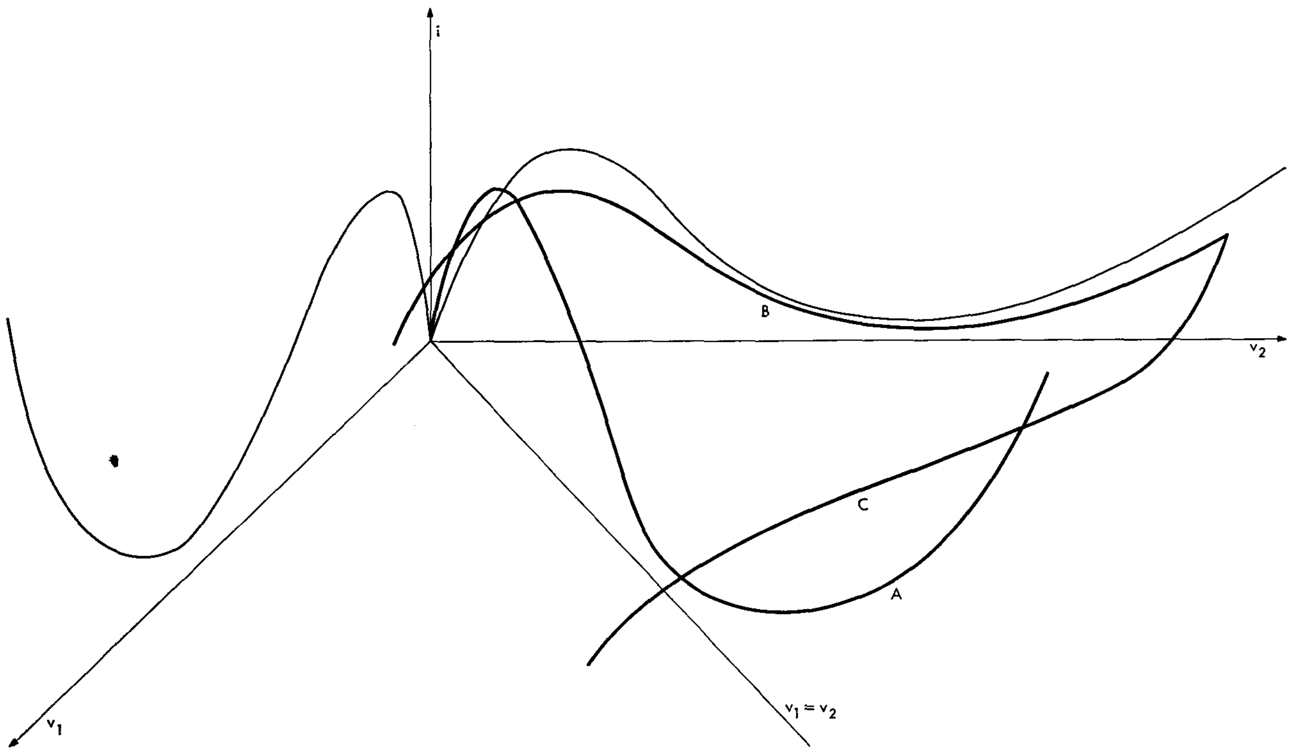
The solutions of these two equations can geometrically be constructed by intersection in a three dimensional i, v_1, v_2 space of the two cylindrical surfaces

$$i=f_1(v_1)$$

$$i=f_2(v_2)\,. \qquad (4.7)$$

Their intersection consists of several pieces of curves which is drawn in Figure 12 under the assumption $f_1(x)=$

Figure 12



$f_2(x)=f(x)$. In this case one branch of the curve (4.7) is

$$v_1=v_2=v \quad ; \quad i=f(v) .$$

But there are two other branches, which bifurcate from the points $v_1=v_2=m$, $i=f(m)$ where m is chosen so that $f'(m)=0$.

Since the curves lie symmetric to $v_1=v_2$ we have indicated only the part in $v_2 \geq v_1$. The curve on $v_1=v_2$ in Fig. 12 is denoted by A while the two other branches are called B, C. The intersection of A and C corresponds to the minimum, that of A and B to the maximum of f.

To solve (4.6a) one has to intersect this curve system A, B, C with the plane

$$i=\frac{E-v_1-v_2}{R}, \tag{4.8}$$

which also lies symmetric to $v_1=v_2$.

One can see easily that for very large values of R the plane (4.8) is almost horizontal and with appropriate choice one can get three intersections of the plane with A, two symmetrical pairs of intersections with B and one symmetrical pair of intersections with C, hence altogether at least $3+2\cdot2+2\cdot1=9$ steady states.[11] This is an undesirable situation which can be avoided by appropriate parameter restrictions.

To ensure that the plane (4.8) and A have only one point of intersection we require

$$-f'(x)<\frac{2}{R}. \tag{4.9}$$

A condition guaranteeing that the plane (4.8) intersects B only in one pair of points is harder to obtain and, in fact, depends on the *shape of the curve* $i=f(v)$. Here we give only one rough condition which shows that the decreasing part of $i=f(v)$ should be slower descending than the increasing part of the curve ascends to the maximum.

Let v_1 and v_2 be determined so that

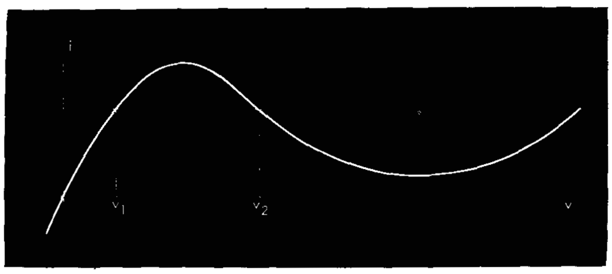
$$f(v_1)=f(v_2) \quad ; \quad v_1<v_2$$

and use v_2 as independent variable, Fig. 13.
Then, differentiating, one has

$$\frac{dv_1}{dv_2}=\frac{f'(v_2)}{f'(v_1)}<0 .$$

The condition, that the branch B pierces the plane from

Figure 13



below, as one moves along B with increasing v_2, amounts to

$$-f'(v_2)<\frac{1}{R}\left(1+\frac{f'(v_2)}{f'(v_1)}\right), \tag{4.10}$$

which is a much more stringent condition than (4.9). In fact, if the curve $i=f(v)$ were symmetric with respect to the line perpendicular to the maximum then $f'(v_1)+f'(v_2)=0$ and (4.10) would be violated.

Since $f'(v_2)<0$ (4.10) can be written in the form

$$R<-\left(\frac{1}{f'(v_1)}+\frac{1}{f'(v_2)}\right)=-\left(\frac{dv_1}{di}+\frac{dv_2}{di}\right) \tag{4.10'}$$

where $f(v_1)=f(v_2)=i$, $v_1<v_2$. One easily computes

$$-\lim\left(\frac{1}{f'(v_1)}+\frac{1}{f'(v_2)}\right)=\frac{4}{3}\cdot\frac{f'''(m)}{f''^2(m)}$$

as v_1, v_2 approach the value m at which f attains its maximum. Therefore (4.10′) implies

$$R<\frac{4}{3}\frac{f'''}{f''^2} \qquad \text{for } v=m . \tag{4.10''}$$

This relation shows that $f'''>0$ at $v=m$ and expresses that f descends slower than it ascends.

To give this condition a more concrete form we discuss the cubic polynomial

$$f(v)=c\left\{\frac{v^3}{3}-\frac{m_1+m_2}{2}v^2+m_1m_2v\right\}$$

when c is a constant of the dimension current/(voltage)^3. f has a maximum at $v=m_1$ and a minimum at $v=m_2$ if $0<m_1<m_2$. In order that $f>0$ for $v>0$ and that the maximum value is about 10 times the minimum value we choose

$$m_2=\frac{29}{10}m_1=\left(3-\frac{1}{10}\right)m_1 .$$

One shows that the condition (4.10′) has to be satisfied only at $v=m_1$ since

$$-\left(\frac{1}{f'(v_1)}+\frac{1}{f'(v_2)}\right)$$

increases as v_2 increases, if $f(v_1)=f(v_2)$. Therefore one has to check only (4.10″) which reads

$$R<\frac{8}{3}c^{-1}\frac{1}{(m_2-m_1)^2}$$

or

$$\frac{1}{R}>\frac{3}{2}\frac{c(m_2-m_1)^2}{4}=\frac{3}{2}\text{Max}\left[-f'(v)\right]$$

or

$$-f'(v)<\frac{2}{3}\frac{1}{R} \qquad \text{for all } v \tag{4.11}$$

in this case. This is a stronger requirement than (4.9),

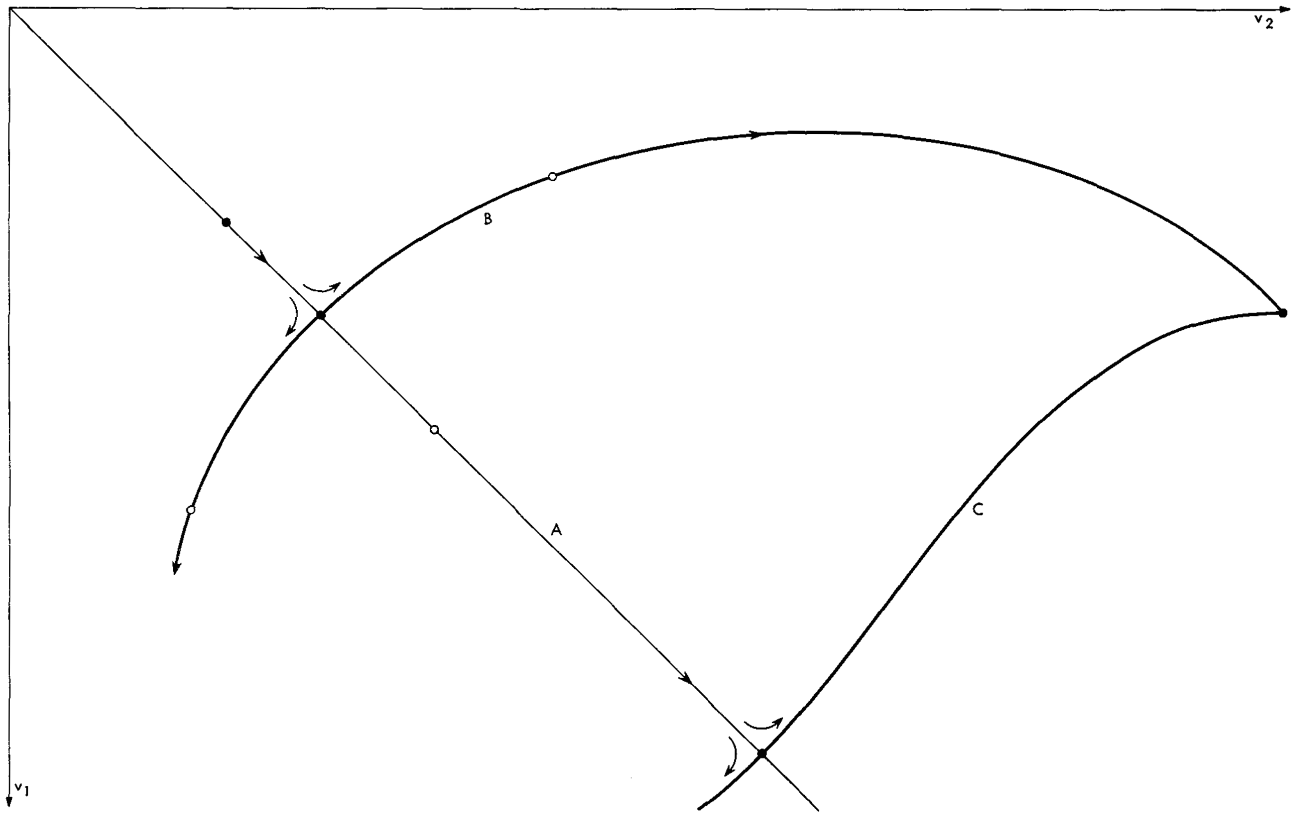


Figure 14

since the right-hand side of (4.9) is multiplied by 1/3.

Under this condition we can have three steady states which lie on the curves A and B. Of course, it can happen that there is only one steady state solution, for instance if $E=0$ in which case the plane (4.8) intersects the curves A, B, C only in $i=v_1=v_2=0$. Ignoring the curve C, since it lies outside the range of interest, we discuss the number of extreme points of U for different values of E.

For sufficiently small $E>0$ there is only one point of intersection between the plane (4.8) and the curves to be considered. The number of extreme points can change only if

$$U_{v_1v_1}U_{v_2v_2}-U^2_{v_1v_2}=\left[\frac{1}{R}+f'(v_1)\right]\left[\frac{1}{R}+f'(v_2)\right]-\frac{1}{R^2}$$
$$=\left(\frac{1}{f'(v_1)}+\frac{1}{f'(v_2)}+R\right)\frac{f'(v_1)f'(v_2)}{R}$$

vanishes. For $v_1=v_2$ and $f'(v)>0$ this expression is positive and vanishes for the first time at the maximum value of f. If we denote this point by $v=m_1$ then the critical value for E is

$$E_m=Rf(m_1)+2m_1 .$$

As E increases beyond E_m one has one steady state on A and a symmetric pair on B. For larger values of E these three points come together again and continue on A. We indicate this behavior of the steady state solutions in Fig. 14, in which A, B, C and the steady states are projected into the v_1-v_2-plane. (At the intersection of A and C there is another bifurcation leading to five steady states.)

Finally we discuss the situation where all three steady states occur, i.e., $E>E_m$ and in which the steady state on A lies on the decreasing branch of f, i.e., $f'(v)<0$.

To discuss the stability behavior of the three solutions we determine at which point U assumes a minimum. Since at any point on A one has

$$\left(\frac{\partial}{\partial v_1}-\frac{\partial}{\partial v_2}\right)^2 U=2f'(v)<0 \qquad (v_1=v_2=v)$$

it is clear that U has no minimum there and the extreme point of U is a saddle point or a maximum. Since U is a symmetric function of v_1, v_2 it follows that the extreme points on B are the minima of U. In fact the three stationary points can be characterized by

$$\operatorname*{Min}_{v_1=v_2} U=U_A$$

and

$$\text{Min } U=U_B .$$

It is clear that the unconditional minimum is smaller than the minimum on $v_1=v_2$, i.e.,

$$U_B<U_A .$$

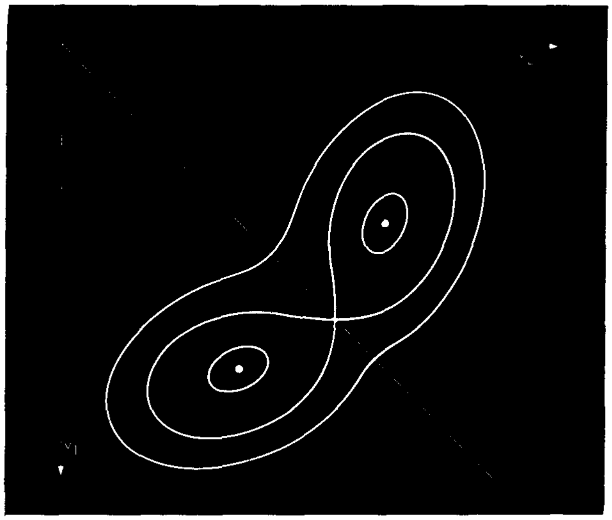


Figure 15

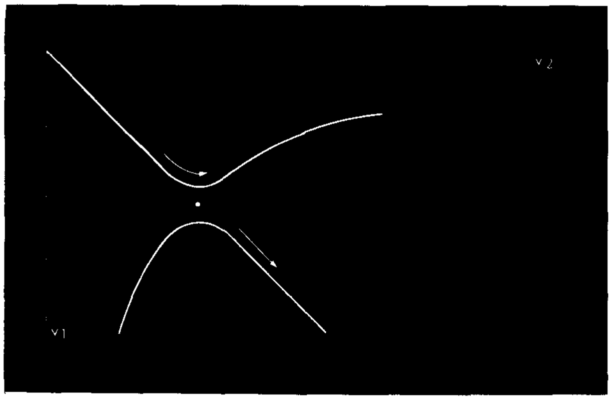


Figure 16

This proves with the Theorems 3.4, 3.6 that the steady states solutions on B are *stable*, provided

$$\frac{f'(v_k)}{C_k}+\frac{R}{L}>0 \qquad (4.12)$$

or

$$\frac{R}{L}>\frac{1}{R}\left(\frac{1}{C_1}+\frac{1}{C_2}\right).$$

However the last condition together with (4.11) implies the first. Therefore it is less restrictive to assume (4.12) only.

For a situation where three steady state solutions exist, the level lines of U will have the appearance as shown in Fig. 15.

- *Summary of twin diode section*

If one assumes that the characteristic $f(v)$ satisfies (4.10′), which for a cubic takes the form

$$-f'(v)<\frac{2}{3}\frac{1}{R}$$

and if

$$-\frac{f'(v)}{C_k}<\frac{R}{L}\ (k=1,2) \text{ or } \frac{R}{L}>\frac{1}{R}\left(\frac{1}{C_1}+\frac{1}{C_2}\right),$$

then all solutions tend to steady state solutions. For small values of E there is only one stable steady state for which $v_1=v_2$. This solution bifurcates at $E=E_m$ into three steady states. The solution with $v_1=v_2$ is unstable while the other two which lie symmetric to the line $v_1=v_2$ are stable. In the latter case most of the energy is in one of the diodes or the other while the case where both diodes have the same energy is unstable if $f'(v)<0$.

- *The influence of a bias*

As was discussed in Section 2, *Twin circuit with bias*, the influence of a bias current E_B, i.e., a signal, is the same as a modification of the characteristic of one of the diodes. Such a bias will destroy the symmetry of U in v_1 and v_2 and we indicate how Figure 14 has to be modified. Figure 16 shows that the stable steady state follows a prescribed curve.

References and footnotes

1. IBM Research Report RC-410, November 17, 1960.
2. IBM Research Report RC-338, September 16, 1960.
3. E. Goto et al, "Esaki Diode High Speed Logical Circuits," *IRE Transactions on Electronic Computers,* **EC-9,** 25-29 (March, 1960).
4. The discussion of the parametron from this point of view (which has been carried out in a previous IBM Report) has been omitted here since it requires quite a different technique.
5. In the meantime it has been shown that the above form of the differential equations is typical for general *RLC* circuits and is not a special feature of these circuits. (See IBM Research Report RC-458, February 23, 1961.) In fact, the function *P* is related to Raleigh's dissipation. For related literature we refer to:
   R. T. Duffin, "Nonlinear Networks IIa," *Bulletin of the American Mathematical Society,* **53,** 963-971 (1947).
   W. Millar, "Some General Theorems for Nonlinear Systems Possessing Resistance," and C. Cherry, "Some General Theorems for Nonlinear Systems Possessing Reactance," *Philosophical Magazine* (ser. 7) **42,** 1150-1160 and 1161-1177 respectively (1951).
6. Lejeune Dirichlet, *Uber die Stabilität des Gleichgewichts,* **22,** (Feb. 1846) (an der königl Akad. der Wiss. gelesen). *L. Dirichlet Werke,* Berlin 1897. Vol. 2, pp. 3-8.
7. More precisely: The variables i and I are related by a Legendre transformation while v_k plays the role of a parameter.
8. See Eq. (3.6).
9. J. Hadamard, *Bull. Soc. Math. de France,* **29,** 224 (1901); see also E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill Book Company, Inc., 1955. Theorem 6.1, p. 387.
10. See also E. A. Coddington and N. Levinson, Ref. 9, Chapter 16.
11. If one replaces $f_1(v_1)$ and $f_2(v_2)$ by cubics the equations (4.6) are algebraic equations with at most nine roots.

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