Pythagorean hodographs

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The *hodograph* of a plane parametric curve $\mathbf{r}(t) = \{x(t), y(t)\}$ is the locus described by the first parametric derivative $\mathbf{r}'(t) = \{x'(t), y'(t)\}$ of that curve. A *polynomial* parametric curve is said to have a *Pythagorean* hodograph if there exists a polynomial $\sigma(t)$ such that $x'^2(t) + y'^2(t) \equiv \sigma^2(t)$, i.e., $(x'(t), y'(t), \sigma(t))$ form a "Pythagorean triple." Although Pythagorean-hodograph curves have fewer degrees of freedom than general polynomial curves of the same degree, they exhibit remarkably attractive properties for practical use. For example, their arc length is expressible as a *polynomial* function of the parameter, and their offsets are *rational* curves. We present a sufficient-and-necessary algebraic characterization of the Pythagorean-hodograph property, analyze its geometric implications in terms of Bernstein-Bézier forms, and survey the useful attributes it entails in various applications.

1. Introduction

The representation of curves and surfaces in a form amenable to efficient, systematic computation is a basic issue in computer-aided design. Those representations that have won widespread acceptance in practical use are almost exclusively *parametric* formulations, based on (piecewise) polynomial functions (see [1] and references therein). Plane curve segments, for example, are usually defined in a form equivalent to

$$x(t) = \sum_{k=0}^{n} a_k t^k,$$

$$y(t) = \sum_{k=0}^{n} b_k t^k \qquad \text{for } t \in [0, 1]. \tag{1}$$

Such segments may be pieced together with various orders of continuity to form *spline* curves for smooth data interpolation; they are easily *rendered* by uniformly incrementing t and evaluating the polynomials (1); and algorithmic procedures are available for computing their intersections (see [2]). An immediate shortcoming of the form (1)—its inability to accommodate conic loci other than the parabola [3]—may be remedied by allowing the *rational* form $\mathbf{r}(t) = \{X(t)/W(t), Y(t)/W(t)\}$, where $X(t)$, $Y(t)$, and $W(t)$ are polynomials. (Obviously, the polynomial curves are a proper subset of the rational curves; the extension to rational forms does not incur any significant computational difficulties—see [1].)

Despite these attractive features, however, polynomial parametric curves have certain inherent limitations that degrade their overall utility in practical design applications. Our aim is to identify a *subset* of the polynomial curves for which these limitations are relaxed, and to highlight the useful properties that ensue. To facilitate this, we appeal to the notion of the *hodograph* of a plane curve $\mathbf{r}(t) = \{x(t), y(t)\}$, i.e., the locus described by the parametric derivative $\mathbf{r}'(t) =$

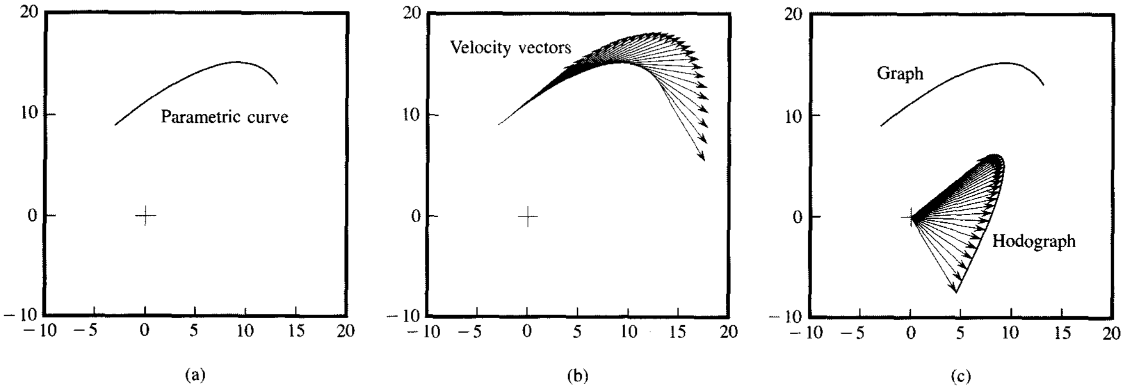

Figure 1

(a) A differentiable parametric curve segment, and (b) sampled derivative or ''velocity'' vectors along that segment. (c) The *hodograph* is the locus generated by translating the continuum of these vectors to the origin.

$\{x'(t), y'(t)\}$ of that curve [4]. If t represents time, the hodograph describes the velocity vector of the trajectory $\mathbf{r}(t)$ (see **Figure 1**). Hodographs are useful, for example, in assessing *a priori* whether two parametric curve segments intersect [5]. Here we are concerned with hodographs of a certain special form, rather than with their use in particular algorithms.

Before proceeding, let us be more specific about some of the shortcomings of polynomial curves alluded to above. When such a curve is rendered by evaluation at a uniform sequence of parameter values $\{t_k\}$, the resulting geometric points $\{\mathbf{r}_k\}$ are *not* uniformly spaced along the curve, since its "parametric flow" is necessarily uneven if it is not merely a straight line. To compensate for this requires a determination of the functional relation between the arc length s along the curve and the parameter t. In general, $s(t)$ is an integral that cannot be resolved into elementary functions of t, and resorting to numerical quadrature to approximate this integral is inefficient and potentially error-prone.

Another problem arises with regard to "offset" curves. In applications such as numerical-control machining, tolerance analysis, and path planning, one is interested in the curve $\mathbf{r}_o(t) = \mathbf{r}(t) + d\,\mathbf{n}(t)$ at a fixed distance d from a given polynomial curve $\mathbf{r}(t)$, in the direction of its unit normal $\mathbf{n}(t)$. The offset $\mathbf{r}_o(t)$ is *not*, in general, a polynomial (or rational) curve. In fact, it was recognized more than a century ago [6] that if $\mathbf{r}(t)$ is of degree n, the offsets at distance $\pm d$ to it, taken together, constitute an irreducible algebraic curve with an implicit equation $f_o(x, y) = 0$ of degree $4n - 2$ in general (see also [7]). This fact has prompted the formulation of several heuristic piecewise-polynomial approximation schemes for offset curves [8–12].

The polynomial curves we identify below overcome these deficiencies. Their arc lengths are merely *polynomial* functions of the parameter, while their offsets at distances $+d$ and $-d$ are individually *rational* curves of relatively low degree (namely, $2n - 1$). In view of the diverse practical uses of offset curves, this latter property is especially significant. Rational forms are the ubiquitous canonical representation scheme of geometric modeling systems, and the possibility of describing offset curves precisely in terms of them facilitates robust processing (rendering, subdivision, transformations, intersections, etc.) of such loci within an existing algorithmic infrastructure.

2. Pythagorean polynomial triples

It is no doubt safe to assume that the reader is familiar with the theorem of Pythagoras,

$$a^2 + b^2 = c^2, \tag{2}$$

relating the length c of the hypotenuse of a right-angle triangle to the lengths a and b of the other sides. Likewise, it is common knowledge that whereas Equation (2) always yields a real value for c when a and b are assigned arbitrary real values, it can be satisfied only in certain special cases—the "Pythagorean triples"—when a, b, and c are *integers* [13].

In this paper we are primarily concerned with the special solutions to an analogous problem, in which the quantities a, b, and c in (2) are (real) *polynomials* in a given variable t, and with the implications of those solutions for the design and processing of parametric curves.

We now recall a particularly simple characterization for the Pythagorean triples of polynomials that will facilitate our investigations:

Theorem (Kubota)
Three real polynomials $a(t)$, $b(t)$, and $c(t)$, where $\max[\deg(a), \deg(b)] = \deg(c) > 0$, satisfy the Pythagorean condition $a^2(t) + b^2(t) \equiv c^2(t)$ if and only if they can be expressed in terms of real polynomials $u(t)$, $v(t)$, and $w(t)$ in the form

$$a(t) = w(t)[u^2(t) - v^2(t)],$$
$$b(t) = 2w(t)u(t)v(t),$$
$$c(t) = w(t)[u^2(t) + v^2(t)]. \quad (3)$$

Proof See Kubota [14]. Although the *sufficiency* of the form (3) for $a(t)$, $b(t)$, and $c(t)$ to satisfy the Pythagorean condition is obvious, its *necessity* is rather subtle. Kubota actually proves this theorem in a far more general context—namely, that of an arbitrary unique factorization domain D of characteristic $p \neq 2$, such that the element 2 is either prime or invertible in D. Here we are concerned exclusively with the case where D is the ring of polynomials in t, over the field of real numbers as coefficients. ■

We assume henceforth that in (3) the polynomials u and v are *relatively prime*, since otherwise the factor $[\text{GCD}(u, v)]^2$ could simply be absorbed into w to render them so. Likewise, we assume that w is a *monic* polynomial, i.e., its leading coefficient is +1, since if that coefficient were $k \neq 1$ we could absorb the constant $\sqrt{k}$ into each of u and v (the assumption that $k > 0$ is justified insofar as the elements (a, b, c) of a Pythagorean triple are considered to be of indeterminate sign).

3. Fundamental aspects of Pythagorean-hodograph curves

The hodograph $\mathbf{r}'(t) = \{x'(t), y'(t)\}$ of a polynomial curve is said to be *Pythagorean* if its components are members of a Pythagorean polynomial triple $(x'(t), y'(t), \sigma(t))$. From the discussion of Section 2 we note that Pythagorean hodographs must be of the form

$$x'(t) = w(t)[u^2(t) - v^2(t)],$$
$$y'(t) = 2w(t)u(t)v(t). \quad (4)$$

(There is no loss of generality in identifying $x'(t)$ with $a(t)$ and $y'(t)$ with $b(t)$ here, since the converse corresponds merely to a rotation of the coordinate axes.) By a "Pythagorean-hodograph curve" we mean any polynomial curve whose derivative is of the form (4).

We begin by dispensing with certain special instances of the hodograph form (4) that are of little practical interest:

a. If either $w(t) = 0$ or $u(t) = v(t) = 0$, Equations (4) reduce to $x'(t) = y'(t) = 0$, and the corresponding real curve locus degenerates to a single point.
b. If $u(t)$, $v(t)$, and $w(t)$ are all constants, and if w and at least one of u and v are nonzero, the real locus defined by Equations (4) is a "uniformly parameterized" straight line, which exhibits the Pythagorean-hodograph property in a trivial sense.
c. If $u(t)$ and $v(t)$ are constants, not both zero, but $w(t)$ is *not* a constant, the real locus given by (4) is again linear (infinite or semi-infinite according to whether $w(t)$ is of even or odd degree), but its *parametric flow* is nonuniform: In fact, it will be "multiply traced" over parameter intervals delineated by the real roots of $w(t)$ of odd multiplicity.
d. Nonuniformly parameterized linear loci can also arise when $w(t) \neq 0$ and either $u(t) = \pm v(t)$ or one of $u(t)$ and $v(t)$ is zero—the former case, which is eliminated by ensuring that $\text{GCD}(u, v) = 1$, yields loci parallel to the y-axis, the latter loci parallel to the x-axis.

Henceforth we shall consider only cases where the polynomials $u(t)$, $v(t)$, and $w(t)$ are all nonzero, $u(t)$ and $v(t)$ being relatively prime and not *both* constants. (These constraints serve merely to eliminate the simpler degenerate forms enumerated above; identifying multiply traced polynomial curves is, in general, a subtle problem [15] beyond our present scope.) The Pythagorean-hodograph curves $\mathbf{r}(t) = \{x(t), y(t)\}$ that satisfy these conditions are necessarily of degree $n = \max[\deg(x), \deg(y)] \geq 3$.

We now examine some of the basic characteristics of Pythagorean-hodograph curves.

Lemma
The polynomial curve corresponding to the Pythagorean hodograph (4) is of degree $n = \lambda + 2\mu + 1$, where $\lambda = \deg(w)$ and $\mu = \max[\deg(u), \deg(v)]$.

Proof On integrating Equations (4), we observe that

$$\deg(x) \leq \deg(w) + 2\max[\deg(u), \deg(v)] + 1, \quad (5a)$$
$$\deg(y) = \deg(w) + \deg(u) + \deg(v) + 1. \quad (5b)$$

We give only a *bound* on $\deg(x)$ because of the possibility of cancellation in the leading terms of $u^2(t) - v^2(t)$. Now if $\deg(u) \neq \deg(v)$, no such cancellation may occur, and $\deg(x) = \lambda + 2\mu + 1 > \deg(y)$, whereas if $\deg(u) = \deg(v)$, we have $\deg(y) = \lambda + 2\mu + 1 \geq \deg(x)$ regardless of whether or not cancellation occurs. Hence

$n = \max[\deg(x), \deg(y)]$ is given by $\lambda + 2\mu + 1$ in all cases. ■

Lemma

Pythagorean-hodograph curves of degree n have (at most) $n + 3$ degrees of freedom, i.e., $n - 1$ fewer than the $2(n + 1)$ degrees of freedom associated with general polynomial curves of the same degree.

Proof If $\mu = \max[\deg(u), \deg(v)]$ (≥ 1 by assumption), the two polynomials $u(t)$ and $v(t)$ are specified by at most $\mu + 1$ coefficients each. If $\lambda = \deg(w)$, however, we associate only λ coefficients with $w(t)$, since this polynomial is assumed to be monic. Thus we may freely choose at most $\lambda + 2(\mu + 1)$ coefficients in specifying the polynomials $u(t)$, $v(t)$, and $w(t)$ that define a Pythagorean hodograph. The constants of integration in (4) yield two further degrees of freedom, making a total of $\lambda + 2\mu + 4 = n + 3$, since $n = \lambda + 2\mu + 1$ by the preceding lemma. ■

These degrees of freedom are not all available for manipulating the *intrinsic shape* of a curve. Three are accounted for in assigning a plane coordinate system (two for choosing an origin and one for orienting the axes), and another two correspond to freedoms in the parameterization, since the curve $\mathbf{r}(\tau)$ resulting from the substitution $t = p\tau + q$ in $\mathbf{r}(t)$ has precisely the same point locus as the latter (q specifies where τ is measured from, while p determines the parametric speed).

Discounting the five freedoms corresponding to rigid motions and reparameterizations (the Pythagorean-hodograph property being invariant under the exercise of these freedoms), we may say that general polynomial curves of degree n enjoy $2n - 3$ "shape freedoms," while Pythagorean-hodograph curves of the same degree have just $n - 2$.

Definition

A polynomial curve $\mathbf{r}(t) = \{x(t), y(t)\}$ has an *irregular point*[1] at each parameter value ξ for which its hodograph traverses the origin, i.e., for which $x'(\xi) = y'(\xi) = 0$.

Evidently the parameter values of the (real) irregular points are the (real) roots of $\phi(t) = \mathrm{GCD}\,(x'(t), y'(t))$. For Pythagorean-hodograph curves, the real irregular points coincide with the real roots of $w(t)$, since it is impossible that $u^2(\xi) - v^2(\xi) = u(\xi)v(\xi) = 0$ for any ξ when $\mathrm{GCD}\,(u(t), v(t)) = 1$.

[1] An irregular point is, of course, a singular point in the usual sense of algebraic geometry (i.e., if $f(x, y) = 0$ is the implicit algebraic equation of $\mathbf{r}(t) = \{x(t), y(t)\}$, then $f_x = f_y = 0$ at that point [16]). However, among the singular points of $f(x, y) = 0$ we must also count its self-intersections, which do *not* (in general) correspond to passages of the hodograph through the origin. A special term is thus desirable to distinguish the latter.

If ξ is just a *simple* root of $\phi(t)$, the irregular point is an ordinary *cusp*, i.e., a point where the curve tangent reverses abruptly. Furthermore, if ξ is of general multiplicity m, then $\mathbf{r}(t)$ will either suffer a sudden tangent reversal or be tangent-continuous at $t = \xi$ according to whether m is odd or even. In the latter case the point $t = \xi$ is still regarded as irregular on $\mathbf{r}(t)$, since the curvature and its derivatives are, in general, unbounded in magnitude there.

The presence of irregular points diminishes somewhat the (global) features of Pythagorean-hodograph curves that are attractive in practical use (see Sections 6 and 7 below). If only finite curve segments are of interest, one can ensure that the chosen parameter domains are devoid of such points. For most applications, however, it is anticipated that the choice $w(t) = 1$ will be adopted and curves constructed from the (relatively prime) polynomials $u(t)$ and $v(t)$ only. Note that the corresponding Pythagorean-hodograph curves are necessarily of odd degree.

We now proceed to a more detailed analysis of the Pythagorean-hodograph property in the context of certain low-degree curves. For this purpose, it is convenient to couch the discussion in terms of the standard Bernstein–Bézier form of a polynomial curve, which affords a numerically stable representation for finite arcs [17]:

$$\mathbf{r}(t) = \sum_{k=0}^{n} \mathbf{p}_k b_k^n(t), \quad \text{where } b_k^n(t) = \binom{n}{k}(1 - t)^{n-k} t^k. \tag{6}$$

The coefficients $\{\mathbf{p}_k\}$ of $\mathbf{r}(t)$ in this representation are known as the "control points" of the curve; they define the vertices of its "control polygon" (see [1] for a review). It is useful to recall some basic properties of the Bernstein basis functions $b_k^n(t)$ in (6), namely, that their indefinite integrals satisfy the relation

$$\int b_k^{n-1}(t)\,dt = \frac{1}{n} \sum_{j=k+1}^{n} b_j^n(t) \quad \text{for } k = 0, 1, \cdots, n - 1 \tag{7}$$

(see [18]), and that they exhibit the *partition-of-unity* property,

$$\sum_{k=0}^{n} b_k^n(t) \equiv 1. \tag{8}$$

The hodograph of the curve (6) may be written in Bernstein–Bézier form as

$$\mathbf{r}'(t) = \sum_{k=0}^{n-1} n\Delta\mathbf{p}_k b_k^{n-1}(t), \tag{9}$$

where $\Delta\mathbf{p}_k$ denotes the kth forward-difference $\mathbf{p}_{k+1} - \mathbf{p}_k$ for $k = 0, \cdots, n - 1$.

It is worth mentioning that while the Bernstein–Bézier form (6) focuses attention on the parameter interval $t \in [0, 1]$, the Pythagorean-hodograph property is fundamentally global in nature. Thus, any constraint on the Bézier control polygon $\{\mathbf{p}_k\}$ over $t \in [0, 1]$ that arises from the Pythagorean-hodograph property must be regarded as applying with equal force to the control polygon over *any* finite parameter span $t \in [a, b]$.

4. Pythagorean-hodograph (Tschirnhausen) cubics

According to the discussion of Section 3, the simplest (nontrivial) Pythagorean-hodograph curves are the cubics with $\lambda = \deg(w) = 0$ and $\mu = \max[\deg(u), \deg(v)] = 1$. These curves have only one shape freedom, as compared to three for the general cubic. We now give a more detailed analysis of these curves, especially with regard to the implications of the Pythagorean-hodograph property for their Bernstein–Bézier forms.

Remark

For a robust construction or verification of the Pythagorean-hodograph property, it is desirable that the coefficients of the polynomials we deal with be specified precisely as elements of the field of rational numbers or an algebraic extension thereof (see the examples below). If they are treated only as floating-point approximations to real numbers, the polynomial nature of the quantity $\sqrt{x'^2(t) + y'^2(t)}$ is almost invariably destroyed.

Consider two linear polynomials $u(t)$ and $v(t)$ given in Bernstein–Bézier form as

$$u(t) = u_0 b_0^1(t) + u_1 b_1^1(t),$$

$$v(t) = v_0 b_0^1(t) + v_1 b_1^1(t), \qquad (10)$$

where we assume that the ratios $u_0 : u_1$ and $v_0 : v_1$ are unequal. The Pythagorean hodograph defined by (10) and $w(t) = 1$ may be expressed as

$$u^2(t) - v^2(t) = (u_0^2 - v_0^2)b_0^2(t) + (u_0u_1 - v_0v_1)b_1^2(t) + (u_1^2 - v_1^2)b_2^2(t), \qquad (11a)$$

$$2u(t)v(t) = 2u_0v_0b_0^2(t) + (u_0v_1 + u_1v_0)b_1^2(t) + 2u_1v_1b_2^2(t). \qquad (11b)$$

In integrating Equations (11), it is convenient to invoke the partition-of-unity property of the Bernstein basis functions and multiply the constants of integration x_0 and y_0 by the left-hand side of (8). Thus, on making use of (7), we may deduce that the Bernstein–Bézier representations of Pythagorean-hodograph cubics,

$$x(t) = \int_0^t u^2(t) - v^2(t)\, dt = \sum_{k=0}^{3} x_k b_k^3(t), \qquad (12a)$$

$$y(t) = \int_0^t 2u(t)v(t)\, dt = \sum_{k=0}^{3} y_k b_k^3(t), \qquad (12b)$$

must have control points $\mathbf{p}_k = (x_k, y_k)$ of the form

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{3}(u_0^2 - v_0^2, 2u_0v_0), \qquad (13a)$$

$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{3}(u_0u_1 - v_0v_1, u_0v_1 + u_1v_0), \qquad (13b)$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{1}{3}(u_1^2 - v_1^2, 2u_1v_1), \qquad (13c)$$

where $\mathbf{p}_0$ is arbitrary, corresponding to the constants of integration. Now the expressions (13) are perhaps not the most palatable characterization of the Pythagorean-hodograph cubics (especially for design engineers). Indeed, we can derive a much more intuitive formulation for these curves in terms of simple geometric parameters describing their control polygons.

Theorem

For a plane cubic $\mathbf{r}(t)$ with Bézier control points $\{\mathbf{p}_k\}$ let L_1, L_2, L_3 be the lengths of the control-polygon legs, and let θ_1, θ_2 be the control-polygon angles at the interior vertices $\mathbf{p}_1, \mathbf{p}_2$. Then the conditions

$$L_2 = \sqrt{L_1L_3} \quad \text{and} \quad \theta_2 = \theta_1 \qquad (14)$$

are sufficient and necessary to ensure that $\mathbf{r}(t)$ has a Pythagorean hodograph.

Proof Let $\mathbf{r}(t)$ be a Pythagorean-hodograph cubic with control points of the form (13), and let d_{jk} denote the distance between $\mathbf{p}_j$ and $\mathbf{p}_k$ ($j \neq k$), so that $L_1 = d_{01}$, $L_2 = d_{12}$, and $L_3 = d_{23}$ (see **Figure 2**). From (13) we see that

$$d_{01} = \frac{u_0^2 + v_0^2}{3},$$

$$d_{12} = \frac{\sqrt{(u_0^2 + v_0^2)(u_1^2 + v_1^2)}}{3},$$

$$d_{23} = \frac{u_1^2 + v_1^2}{3}, \qquad (15)$$

and these expressions clearly imply the first condition $L_2 = \sqrt{L_1L_3}$ given in (14). Further, by the cosine law we may write

$$\cos\theta_1 = \frac{d_{01}^2 + d_{12}^2 - d_{02}^2}{2d_{01}d_{12}},$$

$$\cos\theta_2 = \frac{d_{12}^2 + d_{23}^2 - d_{13}^2}{2d_{12}d_{23}}, \qquad (16)$$

where, according to Equations (13), d_{02} and d_{13} are given by

$$d_{02}^2 = \frac{1}{9}(u_0^2 + v_0^2)[(u_0 + u_1)^2 + (v_0 + v_1)^2],$$

$$d_{13}^2 = \frac{1}{9}(u_1^2 + v_1^2)[(u_0 + u_1)^2 + (v_0 + v_1)^2]. \tag{17}$$

On substituting (15) and (17) into (16) we obtain

$$\cos\theta_1 = \cos\theta_2 = \frac{-(u_0u_1 + v_0v_1)}{\sqrt{(u_0^2 + v_0^2)(u_1^2 + v_1^2)}}, \tag{18}$$

from which we may infer that either $\theta_2 = \theta_1$ or $\theta_2 = 2\pi - \theta_1$. To distinguish between these possibilities, we observe that they imply $\sin\theta_1 = \sin\theta_2$ and $\sin\theta_1 = -\sin\theta_2$, respectively. We may compute the sines of the angles θ_1 and θ_2 as follows:

$$\sin\theta_1 = \frac{(\Delta\mathbf{p}_1 \times \Delta\mathbf{p}_0)\cdot\mathbf{z}}{d_{12}d_{01}},$$

$$\sin\theta_2 = \frac{(\Delta\mathbf{p}_2 \times \Delta\mathbf{p}_1)\cdot\mathbf{z}}{d_{23}d_{12}}, \tag{19}$$

where $\mathbf{z}$ is a unit vector orthogonal to the plane of $\mathbf{r}(t)$. If we substitute Equations (13) and (15) into the expressions (19), we find that

$$\sin\theta_1 = \sin\theta_2 = \frac{u_1v_0 - u_0v_1}{\sqrt{(u_0^2 + v_0^2)(u_1^2 + v_1^2)}}, \tag{20}$$

so that (18) and (20) together imply that $\theta_2 = \theta_1$.

Conversely, let $\mathbf{r}(t)$ be any plane cubic whose control polygon satisfies $\theta_2 = \theta_1$ ($= \theta$, say). We may adopt a coordinate system in which the control-polygon legs have the form

$$\Delta\mathbf{p}_0 = L_1(1, 0),$$

$$\Delta\mathbf{p}_1 = L_2(-\cos\theta, \sin\theta),$$

$$\Delta\mathbf{p}_2 = L_3(\cos 2\theta, -\sin 2\theta), \tag{21}$$

and it is then readily verified that the Bernstein coefficients $\{c_k\}$ of the quartic polynomial $x'^2(t) + y'^2(t)$ are given by

$$c_0 = 9L_1^2, \quad c_1 = -9L_1L_2\cos\theta,$$

$$c_2 = 6L_2^2 + 3L_1L_3\cos 2\theta,$$

$$c_3 = -9L_2L_3\cos\theta, \quad c_4 = 9L_3^2. \tag{22}$$

Thus, if the control polygon of $\mathbf{r}(t)$ also satisfies $L_2 = \sqrt{L_1L_3}$, we find that the coefficients (22) of $x'^2(t) + y'^2(t)$ coincide with those of the perfect square of the quadratic

$$\sigma(t) = 3[L_1b_0^2(t) - L_2\cos\theta\, b_1^2(t) + L_3b_2^2(t)], \tag{23}$$

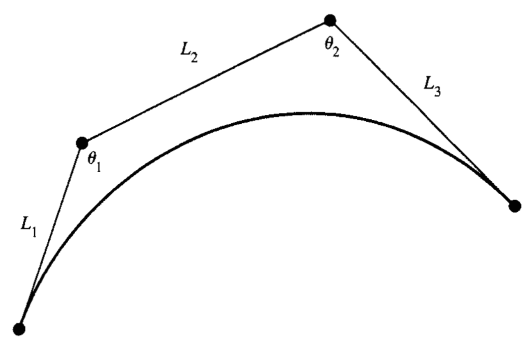

Figure 2

The geometric parameters L_1, L_2, L_3 and θ_1, θ_2 defining the shape of the Bézier control polygon for a plane cubic.

so $\mathbf{r}(t)$ does indeed exhibit a Pythagorean hodograph whenever conditions (14) hold. ■

Recall (Section 3) our earlier remark that Pythagorean-hodograph curves of degree n have just $n - 2$ "shape freedoms." Although we expect the Pythagorean-hodograph cubics to exhibit only one shape freedom, there are, according to (14), apparently three associated with the corresponding Bézier control polygons. Two of the three lengths L_1, L_2, L_3 can be freely chosen, as can the angle θ ($= \theta_1 = \theta_2$). However, two of these freedoms are not essential shape freedoms, being expended by the possibility of reparameterization.

In terms of (u_0, u_1) and (v_0, v_1), we see that the polynomial $\sigma(t)$ that completes the Pythagorean triple with $x'(t)$ and $y'(t)$ given by (11) has the Bernstein–Bézier form

$$\sigma(t) = (u_0^2 + v_0^2)b_0^2(t) + (u_0u_1 + v_0v_1)b_1^2(t)$$
$$+ (u_1^2 + v_1^2)b_2^2(t), \tag{24}$$

which is clearly equivalent to expression (23) given in terms of the geometric parameters L_1, L_2, L_3, and θ.

Examples
The condition $L_2 = \sqrt{L_1L_3}$ implies that the lengths L_1, L_2, and L_3 of the control-polygon legs are either identical or mutually distinct. In examples (a), (b), and (c) below, we have $L_1 = L_2 = L_3 = 1$, while for (d) and (e),

$L_1/2 = L_2 = 2L_3 = 1$ and $2L_1/3 = L_2 = 3L_3/2 = 1$, respectively. These examples are illustrated in **Figure 3**.

(a) $\mathbf{p}_0 = (0, 0)$, $\mathbf{p}_1 = (3/5, 4/5)$, $\mathbf{p}_2 = (8/5, 4/5)$, $\mathbf{p}_3 = (11/5, 0)$.

(b) $\mathbf{p}_0 = (0, 0)$, $\mathbf{p}_1 = (0, 1)$, $\mathbf{p}_2 = (1, 1)$, $\mathbf{p}_3 = (1, 0)$.

(c) $\mathbf{p}_0 = (5/13, 0)$, $\mathbf{p}_1 = (0, 12/13)$, $\mathbf{p}_2 = (1, 12/13)$, $\mathbf{p}_3 = (8/13, 0)$.

(d) $\mathbf{p}_0 = (0, 0)$, $\mathbf{p}_1 = (2, 0)$, $\mathbf{p}_2 = (2, 1)$, $\mathbf{p}_3 = (3/2, 1)$.

(e) $\mathbf{p}_0 = (0, 0)$, $\mathbf{p}_1 = (9/10, 6/5)$, $\mathbf{p}_2 = (19/10, 6/5)$, $\mathbf{p}_3 = (23/10, 2/3)$.

An important common property of these cubic arcs is apparent in Figure 3, namely their *convexity*. This property is, in fact, intrinsic to the Pythagorean-hodograph cubics:

Corollary
Pythagorean-hodograph cubics have no real inflection points.

Proof The absence of inflections on the finite arc $t \in [0, 1]$ follows immediately from the "variation-diminishing" property of the Bernstein–Bézier form (see [1]), since the condition $\theta_2 = \theta_1$ ensures that the control polygons of Pythagorean-hodograph cubics are convex. As noted in Section 3, this feature must generalize to *arbitrary* spans $t \in [a, b]$ of a Pythagorean-hodograph cubic. (Alternately, on substituting the forms (11) for $x'(t)$ and $y'(t)$ into the standard expression $\kappa(t) = [\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{z}/|\mathbf{r}'(t)|^3$ for the curvature, we may observe that the numerator is quadratic in t with discriminant $-4\,(u_0v_1 - u_1v_0)^2$, which is necessarily negative since $u_0 : u_1 \neq v_0 : v_1$ by assumption.) ■

Now it is well known [19] that every plane polynomial (or rational) cubic has a single double point, which may be "at infinity." This double point is necessarily real and is either a *node* or a *cusp*, according to whether the curve exhibits distinct or coincident tangents there. Nodes are further categorized as *crunodes* or *acnodes* according to whether their tangents are real or complex conjugates—the former correspond to self-intersections of the real curve locus, the latter to isolated real points of the curve where conjugate branches of its complex locus cross (see also [20]).

Lemma
Every Pythagorean-hodograph cubic has a crunode, the curve crossing itself at the two distinct real parameter values given by

$$t = \frac{(u_0^2 + v_0^2) - (u_0u_1 + v_0v_1) \pm \sqrt{3}(u_0v_1 - u_1v_0)}{(u_1 - u_0)^2 + (v_1 - v_0)^2} \tag{25}$$

Proof We remark first that, according to the discussion of Section 3, the possibility of an (affine) cusp has been precluded by the choice $w(t) = 1$. Now the double point of a general cubic is identified by parameter values t and $t + \tau$ such that

$$\frac{x(t+\tau) - x(t)}{\tau} = \frac{1}{6}x'''(t)\tau^2 + \frac{1}{2}x''(t)\tau + x'(t) = 0, \tag{26a}$$

$$\frac{y(t+\tau) - y(t)}{\tau} = \frac{1}{6}y'''(t)\tau^2 + \frac{1}{2}y''(t)\tau + y'(t) = 0, \tag{26b}$$

where the division by τ eliminates the trivial solution $\tau = 0$ to $x(t + \tau) - x(t) = 0$ and $y(t + \tau) - y(t) = 0$ for any t. The condition on t such that Equations (26) are simultaneously satisfied for *some* value of τ is given by the vanishing of their resultant

$$R(t) = \text{Resultant}_\tau\left(\frac{x(t+\tau) - x(t)}{\tau}, \frac{y(t+\tau) - y(t)}{\tau}\right) \tag{27}$$

with respect to τ, which may be expressed as the Sylvester determinant [21]:

$$R(t) = \begin{vmatrix} \frac{1}{6}x''' & \frac{1}{2}x'' & x' & 0 \\ 0 & \frac{1}{6}x''' & \frac{1}{2}x'' & x' \\ \frac{1}{6}y''' & \frac{1}{2}y'' & y' & 0 \\ 0 & \frac{1}{6}y''' & \frac{1}{2}y'' & y' \end{vmatrix} \tag{28}$$

It may be verified that, due to a cancellation of leading terms, $R(t)$ is (at most) quadratic in t. Thus, if Δ denotes its discriminant, we may identify $\Delta > 0$ (distinct real roots) with a crunode, $\Delta = 0$ (coincident roots) with a cusp, and $\Delta < 0$ (complex conjugate roots) with an acnode. In particular, when $\mathbf{r}(t)$ has control points of the form (13) corresponding to a Pythagorean hodograph, the resultant (28) assumes the form $R(t) = k[c_2t^2 + 2c_1t + c_0]$, where $k = (u_0v_1 - u_1v_0)^2/9 \neq 0$, and the coefficients c_2, c_1, c_0 are given by

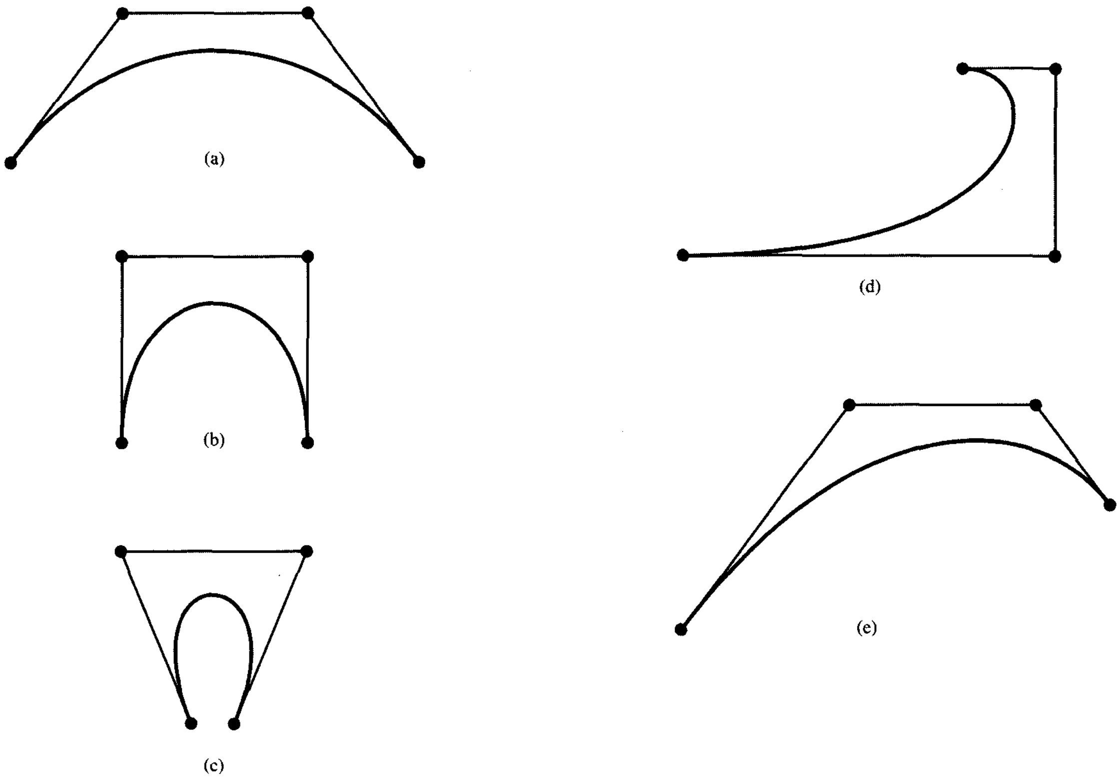

Figure 3

Three Pythagorean-hodograph cubics with symmetric Bézier control polygons (a), (b), (c), and two examples with asymmetric polygons (d), (e).

$$c_2 = [(u_1 - u_0)^2 + (v_1 - v_0)^2]^2, \tag{29a}$$

$$c_1 = [(u_1 - u_0)^2 + (v_1 - v_0)^2][u_0 u_1 + v_0 v_1 - u_0^2 - v_0^2], \tag{29b}$$

$$c_0 = [u_0 u_1 + v_0 v_1 - u_0^2 - v_0^2]^2 - 3(u_0 v_1 - u_1 v_0)^2. \tag{29c}$$

Thus $\Delta = c_1^2 - c_0 c_2 = (4/27)[(u_1 - u_0)^2 + (v_1 - v_0)^2]^2 \cdot (u_0 v_1 - u_1 v_0)^6$, and clearly $\Delta > 0$ if it is assumed that $u_0 v_1 - u_1 v_0 \neq 0$ and $(u_1 - u_0)^2 + (v_1 - v_0)^2 \neq 0$, i.e., that the linear polynomials $u(t)$, $v(t)$ are relatively prime and not *both* constants. The double point is therefore a crunode, and its two parameter values (25) are simply the standard (real) solutions $(-c_1 \pm \sqrt{\Delta})/c_2$ of the quadratic equation $R(t) = 0$. ■

When dealing with finite cubic Bézier arcs, it is usually desirable that Equations (12) define a *simple* curve segment, devoid of self-crossings. This can be guaranteed *a priori* by ensuring that the parameter values (25) do not *both* lie on the interval [0, 1] for the chosen values u_0, u_1 and v_0, v_1.

Now the existence of a crunode is only a *necessary* condition for a cubic to exhibit the Pythagorean-hodograph property. The crunodal cubic $f(x, y) = x^3 - x^2 + y^2 = 0$ [16], for example, admits the parameterization $x(t) = 1 - t^2$, $y(t) = t - t^3$, and in this case we see that $x'^2(t) + y'^2(t) = 9t^4 - 2t^2 + 1 \neq \sigma^2(t)$ for any real polynomial $\sigma(t)$. We now formulate a simple *sufficient* condition for a crunodal cubic to have a Pythagorean hodograph.

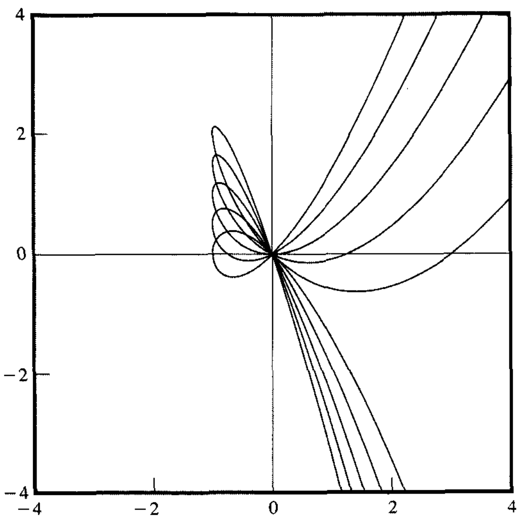

Figure 4

Crunodal plane cubics in standard form, corresponding to a choice of scale factors $p = q = 1$ in Equation (30) and the sequence of values ω = 0.0, 0.5, 1.0, 1.5, 2.0 for the parameter of the x-axis intercept.

Definition
The *standard form* of a crunodal plane cubic $\mathbf{r}(t) = \{x(t), y(t)\}$ is given by

$$x(t) = p(t^2 - 1),$$

$$y(t) = q(t - \omega)(t^2 - 1), \quad (30)$$

which corresponds to a special choice of coordinates and parameterization.

We may interpret the standard form as follows: Taking the double point as origin, we force $x(t)$ and $y(t)$ to possess a common quadratic factor with distinct real roots, corresponding to the two parameter values of the crunode. Now the parameterization may be fixed by assigning parameter values to any two points, so if we take $t = \pm 1$ for the crunode, the common quadratic factor will be $t^2 - 1$. The components of $\mathbf{r}(t)$ then have the form $x(t) = a(t^2 - 1)(t - \alpha)$ and $y(t) = b(t^2 - 1)(t - \beta)$, and a rotation about the origin may be invoked to reduce the factor $t - \alpha$ in $x(t)$ to a constant, giving the form (30). With this orientation, any horizontal line has either one or three real intersections with $\mathbf{r}(t)$, while any vertical line has just zero or two (counted with multiplicity).

Figure 4 illustrates some representative standard-form crunodal cubics. Apart from the independent scale factors p and q for the x- and y-directions, these curves are distinguished by one basic "shape" parameter, namely, the parameter value ω of the x-axis intercept. If $\omega = 0$ the curve is symmetric about the x-axis, whereas it becomes increasingly skewed as $|\omega|$ increases. We now show that the Pythagorean-hodograph property coincides with a special instance of the symmetric case, defined by the ratio $p/q = \sqrt{3}$ of the scale factors.

Theorem
In standard form, the cubic Pythagorean-hodograph curves correspond to instances of the "Tschirnhausen cubic" defined by

$$x(t) = r(t^2 - 1),$$

$$y(t) = \frac{\pm r}{\sqrt{3}}\, t(t^2 - 1). \quad (31)$$

Proof Consider the generic standard-form crunodal cubic (30), for which $x'(t) = 2pt$ and $y'(t) = q(3t^2 - 2\omega t - 1)$. The square of the hodograph magnitude may be written as

$$x'^2(t) + y'^2(t) = q^2[9t^4 - 12\omega t^3 + (4f^2 + 4\omega^2 - 6)t^2 + 4\omega t + 1], \quad (32)$$

where $f = p/q$. If (32) is to be the perfect square of, say, $q(At^2 + Bt + C)$, we must have

$$A^2 = 9, \qquad 2AB = -12\omega,$$

$$2AC + B^2 = 4f^2 + 4\omega^2 - 6, \quad (33a)$$

$$2BC = 4\omega, \qquad C^2 = 1. \quad (33b)$$

The first three conditions (33a) may be regarded as giving the values $A = \pm 3$, $B = \mp 2\omega$, $C = \pm[(2/3)f^2 - 1]$. Enforcing consistency of these values with the last two conditions (33b) then gives constraints on the quantities p, q, and ω for the standard form (30) to exhibit the Pythagorean-hodograph property.

Substituting for B and C into the first equation in (33b), we have $-\omega[(2/3)f^2 - 1] = \omega$, and since $f \neq 0$ by assumption, this can be satisfied only if $\omega = 0$. Further, on substituting for C into the second equation in (33b) we have $(2/3)f^2[(2/3)f^2 - 2] = 0$, which implies that $f^2 = p^2/q^2 = 3$ if $f \neq 0$. Thus $p = \pm\sqrt{3}q$, where the choice of signs corresponds merely to a reversal of the parametric flow. Hence, in standard form, the Pythagorean-hodograph cubics are given by (31), with the quantity $-r$ representing the x-axis intercept. ■

The Tschirnhausen cubic[2] has apparently aroused interest on several occasions, being known also as *l'Hôpital's cubic* and the *trisectrix of Catalan* (see [23] and [24] for further details—however, these references offer no hint of its unique "Pythagorean-hodograph" nature). It is evident from (31) that the single shape freedom of Pythagorean-hodograph cubics corresponds merely to a choice of the uniform scale factor r (see **Figure 5**).

5. Higher-order curves

An important application of parametric cubics is the interpolation of ordered sequences of points in the plane by smooth (C^2) piecewise-cubic curves, i.e., *cubic splines.* The arcs comprising such a spline are usually considered in Hermite form, since the interpolation problem then reduces to solving a tridiagonal system of linear equations for the parametric derivatives at the data points [25]. Unfortunately, the Pythagorean-hodograph cubics are too inflexible for general C^2 interpolation; they cannot interpolate with curvature continuity discrete data whose "shape" implies inflections.

To achieve sufficient flexibility for general free-form design applications while retaining the advantages of Pythagorean hodographs, we must appeal to curves of higher degree. For such curves, however, it would be a difficult and protracted task to provide as complete an analysis as that given in Section 4 for the cubics. Such an analysis would have the following principal aims:

a. To formulate intuitive geometric constraints (such as (14) in the case of cubics) on the control polygon that will guarantee the Pythagorean-hodograph property, or otherwise to provide simple geometric construction procedures for Pythagorean-hodograph curves (i.e., not just substituting chosen polynomials $u(t)$, $v(t)$, $w(t)$ into $x'(t) = w(t)[u^2(t) - v^2(t)]$, $y'(t) = 2w(t)u(t)v(t)$ and integrating).
b. To classify the essential shape freedoms of Pythagorean-hodograph curves of a given degree (e.g., the identification of the cubics with instances of Tschirnhausen's curve), by the analysis of their singular points and the identification of "standard forms," and to assess the suitability of these shape freedoms for use in representative design problems.

For the sake of brevity, we confine ourselves here to just a brief sketch of some of the salient features of quartic and quintic Pythagorean-hodograph curves, and defer a more systematic analysis to a subsequent paper.

According to the arguments of Section 3, the Pythagorean-hodograph quartics are cuspidal curves corresponding to the case $\lambda = \mu = 1$. Thus, it is convenient to write the monic linear polynomial $w(t)$ in the Bernstein–Bézier form

$$w(t) = -\xi b_0^1(t) + (1 - \xi)b_1^1(t), \tag{34}$$

since we can immediately identify $t = \xi$ as the location of the cusp. With $u(t)$ and $v(t)$ as in Equation (10), it may be verified that the Pythagorean-hodograph quartics must have control points of the form

$$\mathbf{p}_1 = \mathbf{p}_0 - \frac{\xi}{4}(u_0^2 - v_0^2,\ 2u_0v_0), \tag{35a}$$

$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{(1-\xi)}{12}(u_0^2 - v_0^2,\ 2u_0v_0) - \frac{\xi}{6}(u_0u_1 - v_0v_1,\ u_0v_1 + u_1v_0), \tag{35b}$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{(1-\xi)}{6}(u_0u_1 - v_0v_1,\ u_0v_1 + u_1v_0) - \frac{\xi}{12}(u_1^2 - v_1^2,\ 2u_1v_1), \tag{35c}$$

$$\mathbf{p}_4 = \mathbf{p}_3 + \frac{(1-\xi)}{4}(u_1^2 - v_1^2,\ 2u_1v_1), \tag{35d}$$

where again the initial point $\mathbf{p}_0$ is arbitrary. Note that the

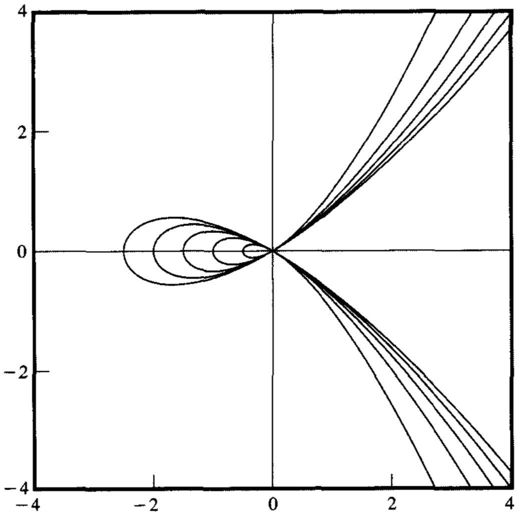

Figure 5

Instances of the Tschirnhausen cubic [Equation (31)] with the values 0.5, 1.0, 1.5, 2.0, 2.5 for the uniform scale parameter r.

[2] Weaver [22] quotes 1690 as the date of Tschirnhausen's identification of this curve; it was also studied by l'Hôpital in 1696.

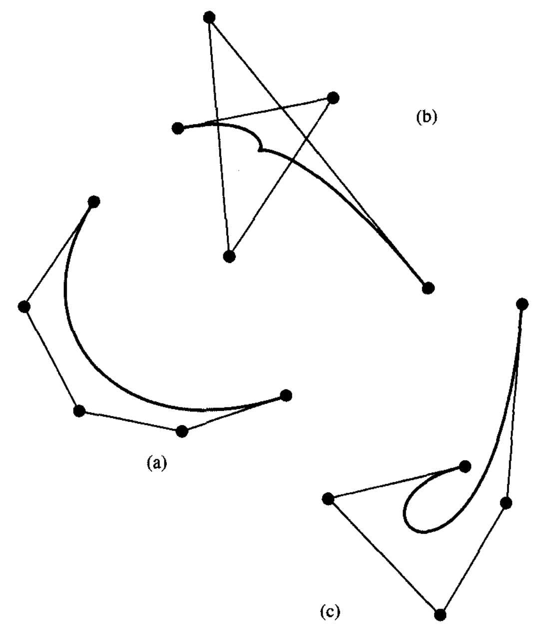

Figure 6

Examples of Pythagorean-hodograph quartics: For (a) and (c) the parameter value ξ of the cusp lies outside the interval [0, 1], whereas for (b) we choose $\xi = 1/2$.

control polygon degenerates if we choose $\xi = 0$ or $\xi = 1$ ($\mathbf{p}_1 = \mathbf{p}_0$ in the former case, and $\mathbf{p}_4 = \mathbf{p}_3$ in the latter). Indeed, this should have been expected, since from (9) we infer that $\mathbf{r}'(0) = 4(\mathbf{p}_1 - \mathbf{p}_0)$ and $\mathbf{r}'(1) = 4(\mathbf{p}_4 - \mathbf{p}_3)$, and we must have $\mathbf{r}'(0) = \mathbf{0}$ or $\mathbf{r}'(1) = \mathbf{0}$ if $t = 0$ or $t = 1$ is a cusp.

Some examples of the Pythagorean-hodograph quartics are illustrated in **Figure 6**. These were generated by making arbitrary choices for the parameters (u_0, u_1), (v_0, v_1), and ξ, and integrating the resulting expressions for $x'(t)$ and $y'(t)$. Obviously, this approach offers little *a priori* insight regarding the shape of the resulting curve.

Figure 6 suggests that the Pythagorean-hodograph quartics might also share the convexity property of the cubics, and indeed it is not difficult to verify that this is the case. The polynomial $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{z}$ is nominally of degree 4 when $\mathbf{r}(t)$ is a quartic, but since $\mathbf{r}'(\xi) = \mathbf{0}$, the cusp incurs a quadratic factor $(t - \xi)^2$ in this polynomial, and the remaining quadratic factor has the discriminant $\Delta = -4(u_0v_1 - u_1v_0)^2$, which is necessarily negative. Thus, the Pythagorean-hodograph quartics have no real inflections.

In the quartic case, the control polygon is described by seven geometric parameters: the lengths L_1, L_2, L_3, L_4 of its four legs and its three interior angles $\theta_1, \theta_2, \theta_3$. The discussion of Section 3 suggests that we should be able to identify from (35) *three* independent constraints on these parameters that characterize the Pythagorean-hodograph property for quartics (note that these quartics have only two shape freedoms, however). Our attempts to generate such constraints proved to be somewhat disappointing, insofar as the resulting equations did not admit as intuitive an interpretation as the conditions (14) for the cubic case. For example, from (35) we deduce the rather obscure condition

$$\begin{aligned} &\xi^2 L_4(3|1 - \xi|L_2^2 - |\xi|L_1L_4) \\ &\quad = (1 - \xi)^2 L_1(3|\xi|L_3^2 - |1 - \xi|L_1L_4) \end{aligned} \tag{36}$$

relating the lengths of the four control-polygon legs and the cusp location ξ. Additional (independent) constraints, involving the control-polygon angles, proved to be even more cumbersome and enigmatic.

Of course, there is no *unique* set of constraints, and more sophisticated analyses (e.g., Gröbner basis reductions) might still reveal a geometrically satisfying set of conditions for the control polygon. The problem of verifying the *sufficiency* of any such conditions for the Pythagorean-hodograph property becomes increasingly difficult as we proceed to higher-order curves, however. It may be that control-polygon constraints are not, in general, a fruitful means of characterizing the higher-order Pythagorean-hodograph curves for practical use; alternate characterizations and/or construction procedures, which offer insight into the curve shape, would then be desirable.

Regarding the shape freedoms of the Pythagorean-hodograph quartics, we observe that since they are rational curves, their singularities must be "equivalent" to three double points [16]. Thus, if we compute the polynomial $R(t)$ defined by (27) for a generic Pythagorean-hodograph quartic, it will be of degree 6. However, we are already aware that ξ must be (at least) a double root of $R(t)$, and the problem thus reduces to characterizing the nature and distribution of the singularities corresponding to the roots of the quartic equation $R(t)/(t - \xi)^2 = 0$, relative to the cusp at $t = \xi$. In principle, this may be achieved by invoking Ferrari's method [21], but since the calculation is quite laborious we do not pursue it here.

The Pythagorean-hodograph quintics are realized by choosing either $\lambda = 0$, $\mu = 2$ or $\lambda = 2$, $\mu = 1$. The curves corresponding to the former case are devoid of irregular points, while those corresponding to the latter have either two ordinary real cusps, one real second-order irregular point, or no real irregular points at all, according to whether the discriminant of $w(t)$ is positive, zero, or negative.

When $w(t)$ is a constant and $u(t)$, $v(t)$ are quadratic, the control points have the form

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{5}(u_0^2 - v_0^2,\ 2u_0v_0), \tag{37a}$$

$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{5}(u_0u_1 - v_0v_1,\ u_0v_1 + u_1v_0), \tag{37b}$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{2}{15}(u_1^2 - v_1^2,\ 2u_1v_1) + \frac{1}{15}(u_0u_2 - v_0v_2,\ u_0v_2 + u_2v_0), \tag{37c}$$

$$\mathbf{p}_4 = \mathbf{p}_3 + \frac{1}{5}(u_1u_2 - v_1v_2,\ u_1v_2 + u_2v_1), \tag{37d}$$

$$\mathbf{p}_5 = \mathbf{p}_4 + \frac{1}{5}(u_2^2 - v_2^2,\ 2u_2v_2). \tag{37e}$$

We mention just one simple constraint on the lengths of the control-polygon legs that arises from expressions (37) in a straightforward manner, namely

$$L_2/L_4 = \sqrt{L_1/L_5}\,. \tag{38}$$

On the other hand, if $w(t)$ is quadratic and $u(t)$, $v(t)$ are linear, the control points become

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{w_0}{5}(u_0^2 - v_0^2,\ 2u_0v_0), \tag{39a}$$

$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{w_0}{10}(u_0u_1 - v_0v_1,\ u_0v_1 + u_1v_0) + \frac{w_1}{10}(u_0^2 - v_0^2,\ 2u_0v_0), \tag{39b}$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{w_0}{30}(u_1^2 - v_1^2,\ 2u_1v_1) + \frac{2w_1}{15}(u_0u_1 - v_0v_1,\ u_0v_1 + u_1v_0) + \frac{w_2}{30}(u_0^2 - v_0^2,\ 2u_0v_0), \tag{39c}$$

$$\mathbf{p}_4 = \mathbf{p}_3 + \frac{w_1}{10}(u_1^2 - v_1^2,\ 2u_1v_1) + \frac{w_2}{10}(u_0u_1 - v_0v_1,\ u_0v_1 + u_1v_0), \tag{39d}$$

$$\mathbf{p}_5 = \mathbf{p}_4 + \frac{w_2}{5}(u_1^2 - v_1^2,\ 2u_1v_1). \tag{39e}$$

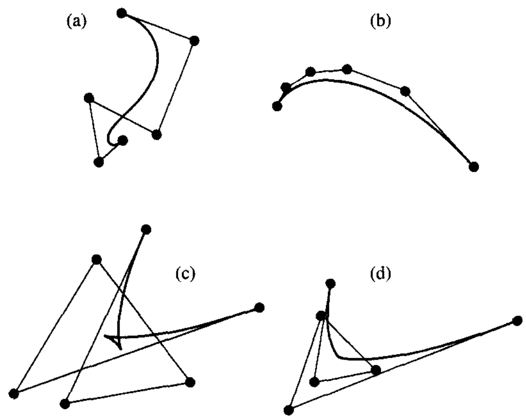

Figure 7

Examples of Pythagorean-hodograph quintics: (a) and (b) correspond to the case $\lambda = 0$, $\mu = 2$, while for (c) and (d) we have $\lambda = 2$, $\mu = 1$. For (c), the roots ξ_1, ξ_2 of $w(t)$ are distinct, giving two cusps, whereas for (d) we set $\xi_1 = \xi_2$, resulting in just one tangent-continuous irregular point of infinite curvature. Note the inflection in (a).

If ξ_1 and ξ_2 are the parameter values of the two cusps, the Bernstein coefficients of $w(t)$ in (39) are given by $w_0 = \xi_1\xi_2$, $w_1 = -[(1 - \xi_1)\xi_2 + (1 - \xi_2)\xi_1]/2$, $w_2 = (1 - \xi_1)(1 - \xi_2)$.

Examples of both the cuspidal and noncuspidal Pythagorean-hodograph quintics are shown in **Figure 7**. Again, these were generated "blind" by freely choosing the parameters (u_0, u_1), (v_0, v_1), and (ξ_1, ξ_2), or (u_0, u_1, u_2) and (v_0, v_1, v_2), as appropriate.

Now the polynomial $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{z}$ is of degree 6 for a quintic, and in the cuspidal case it must contain the factors $(t - \xi_1)^2$ and $(t - \xi_2)^2$, the discriminant of the remaining quadratic term again being $\Delta = -4(u_0v_1 - u_1v_0)^2 < 0$. The cuspidal quintics are therefore necessarily convex, but it is evident from Figure 7(a) that the noncuspidal quintics are the lowest-order Pythagorean-hodograph curves that exhibit real inflections. (When cusps occur, we interpret "convex" to mean that the center of curvature lies consistently to the left or right as we traverse the curve in the sense of its parameterization; see Figure 7(c).)

For quintics, the polynomial (27) is of degree 12 in general, and in the cuspidal case ξ_1 and ξ_2 are (at least) double roots of $R(t)$. Here the analysis of the singular points of Pythagorean-hodograph quintics—and its implications for the shape freedoms of these curves—is more difficult, since we cannot solve by radicals for the parameter values of the singular points, in terms of the coefficients of $u(t)$, $v(t)$, and $w(t)$.

6. Arc length

The arc length along a polynomial curve $\mathbf{r}(t) = \{x(t), y(t)\}$ increases at the rate

$$\frac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t)} \tag{40}$$

with respect to the parameter t. Measuring s from the point $t = 0$, we may write

$$s(t) = \int_0^t \sqrt{x'^2(t) + y'^2(t)}\, dt, \tag{41}$$

but this integral does not, in general, admit a closed-form expression in terms of elementary functions.[3] Computing the arc lengths of polynomial curve segments thus usually entails an approximation by means of numerical quadrature, in specific instances.

If the curve $\mathbf{r}(t)$ has a Pythagorean hodograph, however, there exists a polynomial $\sigma(t)$ such that $x'^2(t) + y'^2(t) = \sigma^2(t)$, so (41) can be rewritten in the form

$$s(t) = \int_0^t |\sigma(t)|\, dt. \tag{42}$$

Indeed, if $\mathbf{r}(t)$ has been constructed by choosing polynomials $u(t)$, $v(t)$, $w(t)$ and integrating the forms (4), we already know that $\sigma(t) = w(t)[u^2(t) + v^2(t)]$. Now the need to take the absolute value of $\sigma(t)$ in evaluating (42) can be a considerable inconvenience, so we consider first those cases where $\sigma(t)$ does not change sign.

Since $\sigma(t) = w(t)[u^2(t) + v^2(t)]$ and GCD $(u, v) = 1$ by assumption, $\sigma(t)$ will have no real roots, and may be assumed positive for $-\infty < t < +\infty$, when $w(t)$ has no real roots. In particular, if $w(t) =$ constant, $\sigma(t)$ will be of degree $n - 1$ and may be written in the form

$$\sigma(t) = \sum_{k=0}^{n-1} \sigma_k t^k > 0 \qquad \text{for } -\infty < t < +\infty, \tag{43}$$

when $\mathbf{r}(t)$ is a Pythagorean-hodograph curve of degree n. The arc length s of $\mathbf{r}(t)$, measured from $t = 0$, is then simply the polynomial function

$$s(t) = \sum_{k=1}^{n} \frac{\sigma_{k-1} t^k}{k} \tag{44}$$

of the parameter t. In this case, $s(t)$ is clearly monotone-increasing with t, since its derivative $\sigma(t)$ is positive for all t.

Example

We content ourselves with a very simple example: the Tschirnhausen cubic (31), for which we have $x'(t) = 2rt$, $y'(t) = \pm r(3t^2 - 1)/\sqrt{3}$, and $\sigma(t) = |r|(3t^2 + 1)/\sqrt{3}$. On integrating, we see that the arc length of this curve, measured from the x-axis intercept, is given by the simple *polynomial* expression

$$s(t) = \frac{|r|}{\sqrt{3}}\, t(t^2 + 1). \tag{45}$$

For a Pythagorean-hodograph curve, finding the parameter value t_0 at which a prescribed total arc length s_0 is attained entails only the determination of the real root of the polynomial equation $s(t) - s_0 = 0$ (where $s(t)$ is given by (44)—note that the monotonicity of $s(t)$ ensures that there is exactly one such root). This should be compared with the problem of determining, by means of numerical quadrature, when the integral (41) attains the desired value s_0 as its upper limit of integration is varied.

Similarly, determining a sequence $\{t_k\}$ of parameter values corresponding to points spaced at uniform arc-length intervals Δs along the curve requires the solution of the sequence of polynomial equations $s(t) - k\Delta s = 0$ for $k = 1, 2, \cdots$ (each of which has a unique real root). If t_k is the solution to the kth equation, it is expected that the expression $t_k + \Delta s/\sigma(t_k)$ will provide an excellent starting approximation for an iterative (e.g., Newton–Raphson) scheme to solve for t_{k+1} when Δs is sufficiently small.

Suppose now that $w(t)$ is *not* a constant. In that case, we need only concern ourselves with the real roots of $w(t)$ of *odd* multiplicity, since it is only at those values that $\sigma(t)$ changes sign (the curve always suffers a tangent reversal at such points). Thus if $t_1, \cdots, t_N$ denote, in ascending order, the real odd-multiplicity roots of $w(t)$, we must break up the integral (42) at those values $\{t_k\}$ that lie within the range of integration and then sum the integrals of $\sigma(t)$ over the resulting subintervals with alternating signs. Clearly, the arc-length computation is more involved in cases where $w(t) \neq$ constant, since it necessitates computing the roots of w.

Finally, it should be noted that if the curve $\mathbf{r}(t)$ is multiply traced over part or all of its real locus (a possibility that cannot easily be eliminated in our construction procedures for Pythagorean-hodograph curves), the arc-length computation will reflect this behavior.

[3] Classically, the problem of determining arc lengths was known as "rectification," and a parametric curve was said to be *rectifiable* if its arc length could be expressed by elementary functions of the parameter [26]. Thus, the family of curves identified in Section 3 might just as well be termed the "rectifiable polynomial curves" (M. A. Sabin, Cambridge, England, personal communication, 1989).

7. Offset curves

If $\mathbf{n}(t)$ is the unit normal vector to a polynomial curve $\mathbf{r}(t) = \{x(t), y(t)\}$ at each point, the *offset* to that curve at (signed) distance d is the locus defined by $\mathbf{r}_o(t) = \mathbf{r}(t) + d\mathbf{n}(t)$. Explicitly, the components of $\mathbf{r}_o(t)$ may be written as

$$x_o(t) = x(t) + \frac{dy'(t)}{\sqrt{x'^2(t) + y'^2(t)}},$$

$$y_o(t) = y(t) - \frac{dx'(t)}{\sqrt{x'^2(t) + y'^2(t)}}. \tag{46}$$

Although Equations (46) constitute a precise description of the offset curve, the presence of the radical $\sqrt{x'^2(t) + y'^2(t)}$ is unfortunate from the perspective of modeling systems that adhere to polynomial and rational forms as their canonical representation. The geometric algorithms of such systems are often crucially dependent on unique attributes of these forms (convergent subdivision algorithms, the variation-diminishing property, etc.), and their robustness may be severely compromised in attempting to accommodate (46) *ad hoc*.

It is possible to describe offsets by *implicit* polynomial equations, if we are prepared to accept representations that simultaneously describe the offsets at distances $+d$ and $-d$ from a given polynomial or rational curve $\mathbf{r}(t)$. For example, the offsets to the parabola $\mathbf{r}(t) = \{t, t^2\}$ constitute an irreducible algebraic curve of degree 6, given by [7]:

$$\begin{aligned} f_o(x, y) = {} & 16x^4(x^2 + y^2) - 8x^2y(5x^2 + 4y^2) \\ & - (48d^2 - 1)x^4 - 32(d^2 - 1)x^2y^2 \\ & + 16y^4 + 2(4d^2 - 1)x^2y - 8(4d^2 + 1)y^3 \\ & + 4d^2(12d^2 - 5)x^2 + (4d^2 - 1)^2y^2 \\ & + 8d^2(4d^2 + 1)y - d^2(4d^2 + 1)^2 = 0. \end{aligned} \tag{47}$$

Equation (47) is actually the *simplest* (nontrivial) implicit equation for the offset to a polynomial curve; in general $f_o(x, y)$ is of degree $4n - 2$ or $6n - 4$, according to whether $\mathbf{r}(t)$ is a polynomial or rational curve of degree n (see [7]).

Considerable attention has recently been devoted to piecewise-polynomial *approximation* schemes for offset curves (see references cited in Section 1). However, such an approach, although perhaps unavoidable in many practical circumstances, is fundamentally alien to the desire for truly robust geometric algorithms. The Pythagorean hodographs identify a family of curves whose offsets may be represented *precisely* in terms of rational forms and are thus fully compatible with the geometric functionality of contemporary modeling systems.

If $\mathbf{r}(t) = \{x(t), y(t)\}$ is a polynomial curve of degree n with a Pythagorean hodograph of the form (4) such that $w(t)$ has no real roots, then $\sigma(t) = \sqrt{x'^2(t) + y'^2(t)}$ must be a polynomial of degree $n - 1$ that is positive for all real t. The offset $\mathbf{r}_o(t)$ at distance d to $\mathbf{r}(t)$ may then be expressed in the rational form $\{X(t)/W(t), Y(t)/W(t)\}$, where

$$X(t) = \sigma(t)x(t) + dy'(t) = \sum_{k=0}^{2n-1} X_k b_k^{2n-1}(t), \tag{48a}$$

$$Y(t) = \sigma(t)y(t) - dx'(t) = \sum_{k=0}^{2n-1} Y_k b_k^{2n-1}(t), \tag{48b}$$

$$W(t) = \sigma(t) = \sum_{k=0}^{2n-1} W_k b_k^{2n-1}(t). \tag{48c}$$

At least one of $X(t)$, $Y(t)$ is of proper degree $2n - 1$, and $W(t)$ is of proper degree $n - 1$ (the latter being expressed in the degree-elevated form (48c) so as to give an explicit description of the offset $\mathbf{r}_o(t)$ in terms of its $2n$ control points $\mathbf{p}_k = (X_k/W_k, Y_k/W_k)$ and associated "weights" W_k).

Now if $\mathbf{r}'(t) = \{x'(t), y'(t)\}$ is expressed in the form (9) and $\sigma(t)$ has Bernstein coefficients $\sigma_0, \cdots, \sigma_{n-1}$, we may invoke the degree-elevation and arithmetic procedures for polynomials in Bernstein form [18] to give the control points of the rational offset curve (48) explicitly as

$$(X_k, Y_k, W_k) = \sum_{j=\max(0,k-n)}^{\min(n-1,k)} \frac{\binom{n}{k-j}\binom{n-1}{j}}{\binom{2n-1}{k}} \times [\sigma_j(x_{k-j}, y_{k-j}, 1) + dn(\Delta y_j, -\Delta x_j, 0)] \tag{49}$$

for $k = 0, \cdots, 2n - 1$.

(As in the case of arc-length computation, one should beware the possibility of real odd-multiplicity roots in $w(t)$. Since they incur a sudden reversal of the normal vector $\mathbf{n}(t)$ to the original curve, we must expect the offset curve to suffer a *point discontinuity* at these parameter values. It is therefore prudent to break up the original curve at the real odd-multiplicity roots $t_1, \cdots, t_N$ of $w(t)$, thereby ensuring that each curve subsegment will have a continuous offset.)

Despite its rather daunting appearance, the formula (49) is not difficult to implement in practice. The offsets to Pythagorean-hodograph cubics, for example, are merely *rational quintics* (an eminently manageable curve form as compared to the general cubic offset—an irreducible algebraic curve $f_o(x, y) = 0$ of degree 10

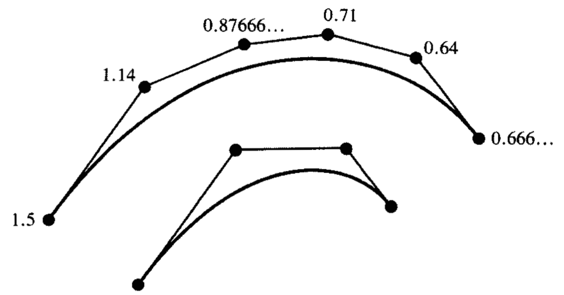

Figure 8

The offset at distance $d = 1$ to the Pythagorean-hodograph cubic of Figure 3(e). The offset curve has a precise rational quintic parameterization; the projective coordinate or "weight" of each Bézier control point is shown.

having 66 terms), and in that case Equations (49) simplify to

$$(X_0, Y_0, W_0) = L_1(x_0, y_0, 1) + d(\Delta y_0, -\Delta x_0, 0), \quad (50a)$$

$$(X_1, Y_1, W_1) = [3L_1(x_1, y_1, 1) - 2L_2 \cos\theta(x_0, y_0, 1) + d(2\Delta y_1 + 3\Delta y_0, -2\Delta x_1 - 3\Delta x_0, 0)]/5, \quad (50b)$$

$$(X_2, Y_2, W_2) = [3L_1(x_2, y_2, 1) - 6L_2 \cos\theta(x_1, y_1, 1) + L_3(x_0, y_0, 1) + d(\Delta y_2 + 6\Delta y_1 + 3\Delta y_0, -\Delta x_2 - 6\Delta x_1 - 3\Delta x_0, 0)]/10, \quad (50c)$$

$$(X_3, Y_3, W_3) = [3L_3(x_1, y_1, 1) - 6L_2 \cos\theta(x_2, y_2, 1) + L_1(x_3, y_3, 1) + d(\Delta y_0 + 6\Delta y_1 + 3\Delta y_2, -\Delta x_0 - 6\Delta x_1 - 3\Delta x_2, 0)]/10, \quad (50d)$$

$$(X_4, Y_4, W_4) = [3L_3(x_2, y_2, 1) - 2L_2 \cos\theta(x_3, y_3, 1) + d(2\Delta y_1 + 3\Delta y_2, -2\Delta x_1 - 3\Delta x_2, 0)]/5, \quad (50e)$$

$$(X_5, Y_5, W_5) = L_3(x_3, y_3, 1) + d(\Delta y_2, -\Delta x_2, 0), \quad (50f)$$

where we use the form (23) for $\sigma(t)$ in terms of the geometric parameters L_1, L_2, L_3, and θ (we drop a common factor 3 above, since an arbitrary scaling may be applied to $X(t)$, $Y(t)$, $W(t)$ without altering the curve). **Figure 8** illustrates the offset to one of the cubic examples of Section 4, constructed according to Equations (50).

8. Concluding remarks

We have not attempted an exhaustive analysis of Pythagorean-hodograph curves here; for higher-order curves, especially, the details are too voluminous for an introductory paper. Our purpose was rather to outline basic defining characteristics, construction procedures, and useful properties for various applications. It is hoped that this will stimulate further study and assessment of the practical utility of these special polynomial curves.

In particular, since the Pythagorean-hodograph quintics appear to enjoy a measure of "shape freedom" similar to that of general cubics, they may constitute a viable alternative to the latter in free-form design applications, affording the attractive attributes discussed in Sections 6 and 7 at the expense of a modest increase in degree. (In Section 5 we mentioned the importance of Hermite forms for the construction of spline curves; in a forthcoming paper [27] we shall show that Pythagorean-hodograph quintic Hermite interpolants exist for *arbitrarily* chosen end points and derivatives of an arc. Furthermore, these interpolants are easily computed and exhibit "shape" properties very similar to those of their standard cubic counterparts.)

The notion of Pythagorean hodographs for plane polynomial curves has straightforward generalizations to other geometric forms that are worthy of detailed investigation. We conclude by briefly outlining a few of these.

Rational curves The rational curve $\mathbf{r}(t) = \{X(t)/W(t), Y(t)/W(t)\}$ has the hodograph

$$x'(t) = \frac{W(t)X'(t) - W'(t)X(t)}{W^2(t)},$$

$$y'(t) = \frac{W(t)Y'(t) - W'(t)Y(t)}{W^2(t)}. \quad (51)$$

(For more on the hodographs of rational curves, see [28].) Here we are interested in those cases where the polynomials $WX' - W'X$ and $WY' - W'Y$ are members of a Pythagorean triple, so that the quantity

$$\frac{ds}{dt} = \frac{\sqrt{(WX' - W'X)^2 + (WY' - W'Y)^2}}{W^2} \quad (52)$$

reduces to a *rational* function of the parameter t. Clearly $WX' - W'X$ and $WY' - W'Y$ must be of the form $w(t)[u^2(t) - v^2(t)]$ and $2w(t)u(t)v(t)$, and the problem is to investigate the implications of these forms for the polynomials $X(t)$, $Y(t)$, $W(t)$ individually, i.e., for the nature of the rational curves that have rational functions of t for ds/dt.

Space curves A twisted polynomial curve $\mathbf{r}(t) = \{x(t), y(t), z(t)\}$ has a three-dimensional hodograph $\mathbf{r}'(t) = \{x'(t), y'(t), z'(t)\}$, and we are interested in the circumstances under which the *three* elements of this hodograph give rise to a polynomial $\sigma(t)$ for the quantity

$$\frac{ds}{dt} = \sqrt{x'^2 + y'^2 + z'^2}. \tag{53}$$

That the hodograph components be expressible in terms of four real polynomials $h(t)$, $u(t)$, $v(t)$, and $w(t)$ in the form

$$x' = h[u^2 - v^2 - w^2],$$

$$y' = 2huv,$$

$$z' = 2huw \tag{54}$$

is evidently a *sufficient* condition, since then $ds/dt = \sigma(t) = h(t)[u^2(t) + v^2(t) + w^2(t)]$; if $h(t)$ is generalized to a rational function it is also *necessary*. In general, one may consider curves of any dimension N, and inquire about the conditions under which the sums of the squares of N polynomials—the hodograph components—coincide with the perfect square of some other polynomial.

Surfaces For a parametric polynomial surface $\mathbf{r}(u, v) = \{x(u, v), y(u, v), z(u, v)\}$, the analog to the quantity $ds/dt = |\mathbf{r}'(t)| = \sqrt{x'^2(t) + y'^2(t)}$ for a plane curve is

$$\frac{\partial^2 A}{\partial u \partial v} = |\mathbf{r}_u \times \mathbf{r}_v|$$

$$= \sqrt{(x_u y_v - x_v y_u)^2 + (y_u z_v - y_v z_u)^2 + (z_u x_v - z_v x_u)^2}, \tag{55}$$

where partial derivatives with respect to u and v are denoted by corresponding subscripts. The integral of (55) over some parametric domain $(u, v) \in \Omega$ gives the corresponding surface area A_Ω, while the surface normal vector

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \tag{56}$$

is unitized by dividing by (55). Thus, we are interested in triples of *bivariate* polynomials $x(u, v)$, $y(u, v)$, $z(u, v)$ such that the argument of the radical in (55) is the perfect square of some other bivariate polynomial $\sigma(u, v)$. If, in addition, we could arrange that $\sigma(u, v) > 0$ over the entire real plane, the offset surface $\mathbf{r}_o(u, v) = \mathbf{r}(u, v) + d\mathbf{n}(u, v)$ would be *rational*, an especially attractive prospect since the problem of reliably approximating offset surfaces [29] is qualitatively more difficult than in the plane curve case.

References

1. G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, New York, 1988.
2. T. W. Sederberg and S. R. Parry, "Comparison of Three Curve Intersection Algorithms," *Comput. Aided Design* **18**, 58–63 (1986).
3. S. S. Abhyankar, "What Is the Difference Between a Parabola and a Hyperbola?" *The Math. Intelligencer* **10**, 36–43 (1988).
4. P. Bézier, *Numerical Control—Mathematics and Applications*, Wiley, New York, 1972.
5. T. W. Sederberg and R. J. Meyers, "Loop Detection in Surface Patch Intersections," *Comput. Aided Geom. Design* **5**, 161–171 (1988).
6. G. Salmon, *A Treatise on the Higher Plane Curves*, Chelsea, New York, 1879 (reprint).
7. R. T. Farouki and C. A. Neff, "Algebraic Properties of Plane Offset Curves," *Comput. Aided Geom. Design* **7**, 101–127 (1990).
8. R. Klass, "An Offset Spline Approximation for Plane Cubic Splines," *Comput. Aided Design* **15**, 297–299 (1983).
9. W. Tiller and E. G. Hanson, "Offsets of Two-Dimensional Profiles," *IEEE Comput. Graph. & Appl.* **4**, 36–46 (1984).
10. J. Hoschek, "Offset Curves in the Plane," *Comput. Aided Design* **17**, 77–82 (1985).
11. J. Hoschek, "Spline Approximation of Offset Curves," *Comput. Aided Geom. Design* **5**, 33–40 (1988).
12. B. Pham, "Offset Approximation of Uniform B-Splines," *Comput. Aided Design* **20**, 471–474 (1988).
13. G. Chrystal, *Algebra, An Elementary Textbook II*, Chelsea, New York, 1889 (reprint).
14. K. K. Kubota, "Pythagorean Triples in Unique Factorization Domains," *Amer. Math. Monthly* **79**, 503–505 (1972).
15. T. W. Sederberg, "Degenerate Parametric Curves," *Comput. Aided Geom. Design* **1**, 301–307 (1984).
16. R. J. Walker, *Algebraic Curves*, Springer-Verlag, New York, 1978.
17. R. T. Farouki and V. T. Rajan, "On the Numerical Condition of Polynomials in Bernstein Form," *Comput. Aided Geom. Design* **4**, 191–216 (1987).
18. R. T. Farouki and V. T. Rajan, "Algorithms for Polynomials in Bernstein Form," *Comput. Aided Geom. Design* **5**, 1–26 (1988).
19. E. J. F. Primrose, *Plane Algebraic Curves*, Macmillan, London, 1955.
20. R. R. Patterson, "Parametric Cubics as Algebraic Curves," *Comput. Aided Geom. Design* **5**, 139–159 (1988).
21. J. V. Uspensky, *Theory of Equations*, McGraw-Hill, New York, 1948.
22. J. H. Weaver, "On the Cubic of Tschirnhausen," *National Math. Magazine* **16**, 371–374 (1942).
23. J. D. Lawrence, *A Catalog of Special Plane Curves*, Dover, New York, 1972.
24. F. Gomes Teixeira, *Traité des Courbes Spéciales Remarquables Planes et Gauches I, II, III*, Chelsea, New York, 1907 (reprint).
25. C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.
26. J. Edwards, *A Treatise on the Integral Calculus*, Chelsea, New York, 1921 (reprint).
27. R. T. Farouki and C. A. Neff, "Hermite Interpolation by Pythagorean-Hodograph Quintics, in preparation (1990).
28. T. W. Sederberg and X. Wang, "Rational Hodographs," *Comput. Aided Geom. Design* **4**, 333–335 (1987).

29. R. T. Farouki, "The Approximation of Non-Degenerate Offset Surfaces," *Comput. Aided Geom. Design* **3**, 15–43 (1986).

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